Notes for Lecture 5
Theory of Matching (Part 3)

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This note answers the question what are optimal reporting strategies for agents on the accepting side in deferred acceptance. It also comments on Nash equilibria, and provides the optional proof of Theorem 1 from Lecture 4.

1 What strategies should the agents on the accepting side follow?

We have seen that the accepting side can sometimes benefit by misrepresenting their preferences. How should they do it?

Consider the exercise from previous lecture. There are four men and four women with the following preferences:

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<thead>
<tr>
<th></th>
<th>$m_1$</th>
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<th>$m_3$</th>
<th>$m_4$</th>
<th>$w_1$</th>
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<tbody>
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<td>$m_2$</td>
<td>$m_1$</td>
<td>$m_4$</td>
<td>$m_3$</td>
<td>$w_4$</td>
</tr>
</tbody>
</table>

Notice that when $w_1$ truncates her preference ranking and reports that only
is acceptable then in the men-proposing DA, $w_1$ is matched with $m_3$, her top choice, and a better match than $m_4$ she would be matched with if she reported her true preference ranking. Notice also that $m_3$ is woman $w_1$ match in women-proposing DA.

This exercise/example illustrates a general phenomenon. To formulate let it, let us say that an agent truncates her or his preference ranking if he or she picks a target acceptable partner and reports that only partners at least as good as the target are acceptable and they are ranked according to true preferences.

**Theorem 1.** Fix reports of all agents but one. If this agent is on the accepting side of DA then an optimal report is to truncate her or his preferences at the best possible outcome in a stable matching.

We have seen in Lecture 4 that an agent’s best possible partner in a stable matching can be found by running deferred acceptance in which the agent is on the proposing side. Taken together, these two results give us a general approach to finding optimal strategies of agents on the accepting side.

For instance, consider an economy with six (or more) men in which woman $w$ has preferences $m_1 > m_2 > m_3 > m_4 > m_5 > m_6 > u$. If there are three stable matchings in which $w$ is matched, respectively, with $m_3, m_4,,$ and $m_6$ then an optimal strategy for her is to report preference profile $m_1 > m_2 > m_3 > u$.

As an aside, let me note that the above truncation remains an optimal strategy for an agent on the accepting side when some of the other agents on the accepting side play such truncation strategies (and other agents continue to report truthfully). You can find the analysis of the above theorem in David Gale and Marilda Sotomayor “Ms. Machiavelli and the stable matching problem,” *American Mathematical Monthly*, 92:261-268, 1985.
2 Nash Equilibria

The above truncation strategies for the accepting side, together with truthful reporting by the proposing side form a Nash equilibrium. A profile of strategies is in Nash equilibrium if each agent $i$ plays an optimal strategy given the strategies of others (that is no other strategy gives $i$ a strictly better outcome than the Nash equilibrium strategy provided other agents play the Nash equilibrium strategies).

As an optional exercise, you may want to check whether there are other Nash equilibria.

3 Optional Proof of Theorem 1 from Lecture 4

Recall that Theorem 1 from Lecture 4 established that there is no other stable matching where man $m$ is married to woman whom he prefers over his match in the men-proposing deferred acceptance.

The proof of this theorem is optional (as all proofs of lecture results). To prove this theorem, first note that it is enough to prove the following claim: if a man $m$ is rejected by a woman $w$ in a round of DA then no stable matching has them matched to each other. We will use the induction principle to prove this claim. The claim is clearly true before the algorithm starts as there are no rejected men then. Suppose this is the case through round $t$; we need to show that this is also the case in round $t + 1$. Assume that at round $t + 1$, woman $w$ rejects man $m$. Note that she either does it because $m$ is unacceptable to her (in which case they cannot be matched to each other in a stable matching), or because she some other man $\tilde{m}$ proposed

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1 This is the same concept of Nash equilibrium you encountered in courses such as Econ 101.

2 This theorem also says that if a man is unmatched in the outcome of men-proposing DA, then this man is weakly worse off (hence also unmatched) in all other stable matchings. This claim can be established by an analogous argument to the one given here.
to her in round $t + 1$ and she prefers $m'$ over $m$. Let us analyze the latter case.

Can there be a stable match that includes $(m, w)$? By way of contradiction, consider the case that there is such a matching. Notice that in this matching $m$ must be matched with a woman $w'$ whom he prefers to $w$ as otherwise $m'$ and $w$ would form a blocking pair. But then man $m'$ would have proposed to $w'$ before he proposed to $w$ in round $t + 1$; since she proposed to $w$ in round $t + 1$ woman $w'$ must have rejected him in one of the earlier rounds. But, then the inductive assumption tells us $m'$ and $w'$ cannot be together in a stable matching — a contradiction that shows that $m$ and $w$ cannot be together in a stable matching. This ends the proof of the theorem.