1. The allocation is efficient since nobody can be strictly better off without making somebody worse off. First note that s1 and s2 have their top choice and hence cannot improve. Now consider s4. In order to improve s4’s outcome she has to be matched with school C, but that would make s2 strictly worse off, so the outcome would not be a Pareto improvement. Finally, note that s3 could be better off only if she is matched with schools C or B. However, giving school C to s3 would make s2 strictly worse off as before and giving school B to s3 would make s4 worse off because we cannot match her with C without hurting s2.

Alternative answer: The outcome can be achieved if one runs serial dictatorship with an order of (s1, s2, s4, s3). By Theorem 1 of lecture 8, this outcome is efficient.

2. a) The resulting allocation is {(s1, A), (s2, C), (s3, B), (s4, D)}. Student s1 will pick school A in the first round, which is her top choice. Then s2 will pick school since it is her top choice and is still available. Student s3 will pick B, which is also her top choice and is available. Finally s4 will pick D since is the only available school and she prefers it over having no school.

b) The allocation is efficient; nobody can be better off without making someone strictly worse off: First, we see that s1, s2, and s3 get their top choice; hence they cannot improve with a different allocation. Also, s4 would be better off if we assign her school B or C, but that would make either s3 or s2 worse off, respectively.

Alternative answer: This is a direct implication of Theorem 1 of lecture 8.

c) Student s2 cannot improve his allocation by submitting a different preference, because student s2 already got the top choice by submitting the true one.

Alternative answer: By Theorem 1 of lecture 8, serial dictatorship is strategy-proof, thus, he cannot get a better match by lying.

d) All students except s4 are assigned to their top choices. s4 receives his neighborhood school. Therefore, everyone receives an assignment as good as their neighborhood school.

3. a) The DA leads to the following allocation {(s1, A), (s2, C), (s3, B), (s4, D)}

b) The allocation is efficient; nobody can be better off without making someone strictly worse off: First, we see that s1, s2, and s3 get their top choice; hence they cannot improve with a different allocation. Also, s4 would be better off if we assign her school B or C, but that would make either s3 or s2 worse off, respectively.
c) Student s2 cannot improve her allocation by submitting a different preference ranking since deferred acceptance with student's proposing is strategy-proof for students (Theorem 3, Lecture 4).

Alternative answer: Student s2 cannot improve his allocation by submitting a different preference, because student s2 already got the top choice by submitting the true one.

d) All students except s4 are assigned to their top choice. s4 receives his neighborhood school. Therefore, everyone receives an assignment as good as their neighborhood school.

4. a) Round 1: s1 points to A. s2 and s4 point to C. s3 points to B. Each School points to the student who is endowed with it. There are two cycles (s1 → A → s1) and (s2 → C → s3 → B → s2) We assign s1, s2, and s3 to A, C, and B, respectively.

Round 2: s4 points to D. School D points to s4. There is one cycle (s4 → D → s4). We assign s4 to D.

Final outcome is \{(s1, A), (s2, C), (s3, B), (s4, D)\}

b) The final outcome is efficient because it is the same allocation in Q2. We also know that top trading cycles is Pareto efficient (Theorem 2, Lecture 8).

c) Student s1 cannot improve her allocation since she is already matched with her top choice. We also know that top trading cycles is a strategy-proof mechanism (Theorem 2, Lecture 8).

d) Students are endowed with their neighborhood schools. By Theorem 2 on lecture 8, every student gets a school at least as good as their endowment.

Alternative Answer: s2 and s3 get better outcome, and s1 and s4 get the original endowment.

5. a) The allocation is calculated in b) below.

b) The TTC algorithm runs as follows:
Round 1: s1 points to A. s2 and s4 point to C. s3 points to B. Each School points to the student who is endowed with it. There is one cycle (s3 → B → s4 → C → s3). We assign s3 and s4 to B and C respectively.

Round 2: s1 still points to A. s2 points to D because C is already assigned to s4. Each School points to the student who is endowed with it. There is one cycle (s1 → A → s2 → D → s1). We assign s1 and s2 to A and D, respectively.

Final outcome is \{(s1,A), (s2,D), (s3,B), (s4,C)\}

c) The answer is no. s2 receives D, which is worse than his neighborhood school B.