1 Solving Nonlinear Models

Solving dynamic nonlinear models is complicated because in almost all cases, a closed-form solution does not exist. I will define a closed form solution as one in which the date–t control variables of the model can be solved for by inverting a functional relationship in which $y$ are model variables, $u$ are shocks, and $\beta$ are model parameters:

\[
H(E_t(y_t, y_{t+1}, u_t, u_{t+1}, \beta)) = 0
\]

\[
y_t = J(E_t(y_{t+1}, u_t, u_{t+1}, \beta))
\]

Solving dynamic nonlinear models is complicated because the solution typically involves a nonlinear function of an expectation of future variables, which themselves may be nested within a nonlinear function. In the expression above, solving for $y_t$ means evaluating a nonlinear function of expected future control variables and expected future shocks. Two examples in which this can be done are models with (i) quadratic objective functions and linear constraints, and (ii) log-linear objective functions and log-linear constraints.

**Exercise:** Show that the one sector growth model with CRRA utility, Cobb-Douglas technology, and complete depreciation of capital has a closed-form solution.

In cases which don’t have closed forms, there are three widely discussed approaches:

- Perturbation methods (Local approximation: First order Taylor series expansion around a steady state, Second order expansion, etc.)
- Projection methods (Global approximation: Finite element)
- Stochastic simulation methods (Parameterized expectations)

It is generally regarded that the details of the specific problem typically dictate which procedure might be best. For example, perturbation methods are fast and can accomodate many state variables. However, they are a local approximation, and typically require differentiable problems, and feature constraints that either always bind, or that never bind. The stochastic optimal growth model can be easily and quickly solved using perturbation methods and with a high level of accuracy, assuming that risk aversion is not very high. Perturbation methods are typically the first approach considered because of their
speed. When it is deemed necessary to move away from a local approximation, or when there are non-differentiabilities or other features of the problem that indicate that perturbation methods may not be adequate, the alternatives are projection methods and stochastic simulation methods. Projection methods are slower and typically don’t accommodate a high dimension state space. They also may not approximate local behavior as well as local method. Below, we consider a stochastic simulation method, parameterized expectations, that has been used widely over the last 25 years. Note that almost always, numerical solutions of these models are challenging and often involve trial-and-error approaches in order to obtain a solution. To be able to successfully carry out some of the projection methods requires a significant investment in time and experience.


**Parameterized Expectations Algorithm (PEA)**

The idea: replace nonlinear functions of expectations of random variables with a known function that only involves known (date t) state variables. This means that we need to come up with a function that approximates the unknown nonlinear function of expectations closely. This is perhaps the most widely used algorithm in cases in which perturbation methods are not considered adequate. Like all numerical approximations, it can be challenging, but it is (in my opinion) more straightforward to use than many of the projection methods.

PEA exploits relationships from dynamic nonlinear models that have the following property

\[ g(E_t[\phi(z_{t+1}, z_t)], z_t, z_{t-1}, u_t) = 0 \]  

(3)

\( g \) is an invertible function

\( z_t \) is a vector of model variables

\( u_t \) is a vector of shocks

Consider solutions in which the expectation is a stationary function of state variables

\[ E_t[\phi(z_{t+1}, z_t)] = E[\phi(z_{t+1}, z_t) | x_t] \]  

(4)

where \( x \) are the state variables and are a subset of \( (z_{t-1}, u_t) \). For example, in the stochastic optimal growth model with a Markov productivity shock (e.g.
Hansen (1985)), the state variables for the planner’s solution are the current value of the productivity shock and the capital stock.

We search for a stationary function:

\[ F(x_t) = E_t[\phi(z_{t+1}, z_t)] \]  \(5\)

Once we have \( F(x_t) \), we can obtain an approximate solution to the model. Again, we typically don’t know \( F(x_t) \), but we know that it should have some characteristics. We know that it should be accurate - that is, it should be able to forecast its right hand side accurately relative to other forecasting rules. This means that it should have a low mean square forecast error and that the forecast errors should be white noise.

Note: we will need a solution in which \( G() \) is invertible with respect to its second argument.

The paper by Lorenzoni and Marcet describes several examples of model economies to identify the objects \( z(x_t) \). We will turn to those now. Note that my notation of \( z(x_t) \) differs from that in Lorenzoni and Marcet because the character they use is not available (to my knowledge) as a latex object.

### 1.1 Method

Approximate the conditional expectation function that resides within the function \( G() \) with a function of state variables. Call this approximating function:

\[ \psi(x_t; \beta) \]  \(6\)

Recall that \( x_t \) is a vector of date \( t \) state variables, and \( \beta \) is a vector of coefficients. Assume that \( \beta \in \mathbb{R}^k \). Next, plug this function into our original \( G \) function as follows:

\[ g(\psi(x_t(\beta); \beta), z_t(\beta), z_{t-1}(\beta), u_t) = 0 \]  \(7\)

Note we still have not specified a function or assigned values to coefficients.

(1) Write the system \( g() \) so that it is invertible with respect to its second argument. The system \( g \) has to have the feature that given the first, third and fourth arguments the value for \( z_t \) can be be uniquely determined by inverting this function. Find a set of state variables, \( x \) that are sufficient for forecasting the conditional expectation that we want to approximate. Replace the true conditional expectation with the approximating function. Fix the initial conditions for the shocks, \( u_0 \) and \( z_0 \). Draw a sequence of \( u_t \) from a (pseudo) random number generator for a long sequence. The \( u \) are drawn according to their distribution. Thus, if you specify that \( u \) is Guassian, then you draw (pseudo) random numbers from a Guassian distribution, etc.
(2) For a specified $\beta$ (the elements of $\beta$ have numbers assigned to them), recursively calculate $z_t(\beta)$, given $\{u_t\}_{t=0}^T$. Note that we require the values of $\beta$ to be such that the resulting $\{z_t\}$ are stationary.

(3) Find the minimizer of the following expression:

$$
G(\beta) = \arg \min_{\xi \in \mathbb{R}^k} \frac{1}{T} \sum_{t=0}^{T} \| \phi(z_{t+1}(\beta), z_t(\beta)) - \psi(\xi; x_t(\beta)) \|^2
$$

This minimization is the key to parameterized expectations. It says the following: choose a coefficient vector, $\xi$ such that the average squared deviation between the realized values of $\phi(z_{t+1}(\beta), z_t(\beta))$ and the approximating conditional expectation $\psi(\xi; x_t(\beta))$ is minimized. Given quadratic loss, we minimize this expression using nonlinear least squares. The left hand side variable in the regression is $\phi(z_{t+1}(\beta), z_t(\beta))$ and the right hand side variable is $\psi(\xi; x_t(\beta))$, and we solve for $\xi$ to minimize the expression.

Marcet and Marshall (1994) discuss the convergence properties of this iteration.

We iterate on the above step by updating the value of $\beta$. We choose a linear combination of the previous value of $\beta$ and the regression coefficients, $\xi$.

$$
\beta_{i+1} = \omega \beta_i + (1 - \omega)\xi
$$

(4) Find the fixed point of $G(\beta)$, $\beta_f = G(\beta_f)$. This is done by iterating on the above step until $\beta$ stops changing. To obtain convergence, we need to start with an initial guess for $\beta$ such that the simulated series do not diverge. Maliar and Maliar (2003) discuss how to do this and we will also cover this.

Then, we have an approximation to the conditional expectation in our $G(\cdot)$ function. More specifically, given $\{u_t\}$ and given the functional form for $\psi$, we have obtained the best (minimum MSE ) predictor of the conditional expectation in $G(\cdot)$. It is the best predictor in the sense that any other $\beta$ will yield a higher MSE. You can now see why we draw a long sequence of $u$, and you can also now see why this is stochastic simulation solution method. Next, given that we have the best approximation to the conditional expectation, we then invert $G$ to calculate the date $t$ control variables. At this point, we are done!

Cautionary notes:

Researchers tend to have trouble for two reasons

(1) A poor initial guess. To get around this, use a homotopy in which we start with the case in which the solution to the problem is known (e.g. the
growth model with complete depreciation), and then we slowly move away from the closed form solution case. Alternatively, one can start with coefficient values that have been obtained by other researchers.

(2) Numerical instability in the updating component. This can happen when the approximating polynomial is high order, which in turn may induce collinearity. In this case, one can use approaches such as principal components or singular value decomposition. (See Den Hann, and Maliar and Maliar).

Next, we go through the examples in Lorenzoni and Marcet.