This note introduces the model of auctions of a single indivisible object. (1) The basic auction model in the symmetric complete information case (bidders know each others values), and examines second-price sealed bid auctions and first-price sealed bid auctions in this context.

1 Baseline Model

We consider a setting with one good (or object) to be auctioned and \( n \) bidders, each of whom would like to obtain it. Each bidder \( i = 1, ..., n \) has value \( v_i \) for the auctioned good. Over the course of our class meetings we will consider various scenarios of what the bidders know about their and other bidders’ values.

The bidders participate in an auction. We will consider several auction formats. In each of the formats the bidders submit bids; we focus on auctions in which each bid comes from some set of allowable bids \( G \subseteq [0, +\infty) \). In all auctions we study, a bidder who submitted the highest bid wins the good. If several bidders submitted the same highest bid, the good will be allocated to one of these highest bidders at random; we assume that each one of them will have equal chance of obtaining the good (that is winning the
auction). The bidders then make payments to the auctioneer; we will focus on auctions in which only the winning bidder pays. Auctions satisfying the above assumptions are commonly referred to as conventional auctions; the two leading examples of such auctions are first-price auctions and second-price auctions discussed below.

If bidder \( i \) wins the good and pays price \( t \) then his or her profit is \( v_i - t \). If the bidder does not win the good and pays nothing then her or his profit is 0. If the bidder wins with probability \( p \) (for instance because of tie-breaking) and pays \( t \) when winning then this bidder’s expected payoff is \( p(v_i - t) \). We assume throughout that each bidder wants to maximize his or her expected payoff/profit (we will interchangeably refer to a bidder’s profit as payoff).

2 Auctions in Symmetric Complete Information Environment

We first consider the model in which each bidder knows her or his value as well as the values of others. We will check whether auctions are strategy-proof and we will also analyze Nash equilibria in auctions. We studied both of these concepts in previous parts of the course.

We will also check whether auctions are strategy-proof. As in the lectures on matching and no-transfer allocation, a mechanism is strategy-proof if revealing the true value is a dominant strategy; in other words bidding one’s true value is at least as good as any other bid irrespective of bids of others.

Let us now look at several canonical examples of auctions assuming that all bids in \( G = [0, +\infty) \) are allowed.

2.1 Second-Price Sealed-Bid Auction

The auction proceeds as follows:

- Bidders ‘simultaneously’ submit sealed bids.
• Bidder who submitted the highest bid wins the good
• Winner pays the second highest bid.

If two or more bidders submitted the same bid, and this is the highest bid, then we need some additional rule to decide who gets the object; such a rule is called the tie-breaking rule. As mentioned above, we assume that in case of such a tie, the object is allocated at random to bidders who submitted the highest bid and that each of these bidders has the same probability of getting the object.

Consider the following

Exercise 1. There are two bidders. Bidder 1’s value is 1 and bidder 2’s value is 1.

How should one bid? Is bidding one’s true value a dominant strategy? What are Nash equilibria?

Following the long tradition in economics we will address this question by finding a Nash equilibrium. In the second-price auction things are simple as it turns out that bidding one’s true value, \( b_i = v_i \) is a dominant strategy — a bidder wants to do it, no matter what others are doing (recall that we talked about dominant strategies in the context of matching and no-transfer allocation). In particular, bidding my true value maximizes a bidder’s payoff if other bidders bid their true values; that is bidding true value is a Nash equilibrium.

Theorem 1. In the second price auction bidding one’s true value is a dominant strategy, and a Nash equilibrium.

This theorem implies that both bidders bidding \( b_1 = b_2 = 1 \) is an equilibrium of the second-price auction in Exercise 1. There are other equilibria.

\(^1\)In our analysis of the second-price auction, tie-breaking will not play much of a role (in well-optimized second-price auction if there is a tie, bidders are indifferent whether they win or not). Tie-breaking plays a more important role in the first-price auctions discussed next.
For instance, \( b_1 = 1 \) and \( b_2 = 2 \) is an equilibrium as is \( b_1 = 3 \) and \( b_2 = 0 \). On the other hand, \( b_1 = b_2 = 3 \) is not an equilibrium as either of the two bidders is better off by bidding less.

**Proof of Theorem 1.** We want to show that bidding your true value \( v \) is at least as good as any other bid \( b \). First let us check that you cannot be better off by bidding \( b > v \) than by bidding your value \( v \).

- If the highest opposing bid is less than \( v \), or higher than \( b \), it makes no difference whether you bid \( b \) or \( v \).

- If the highest opposing bid is between \( v \) and \( b \); you win if you bid \( b \) while you would lose bidding \( v \), but you are better off losing in this case as by winning you pay the second highest bid which is more than your value.

Second, let us check that you cannot be better off by bidding \( b < v \) than by bidding your value \( v \). As before, the outcome of the two bids only differs if the highest opposing bid is between \( b \) and \( v \). And, in this case bidding \( v \) is better as you win and pay less than your value while you would lose by bidding \( b \).

While the above theorem identifies one Nash equilibrium of the second-price auction, there might be others. For instance, in example 1, bidder 1 bidding 0 and bidder 2 bidding 1 is another Nash equilibrium. In the same example, bidder 1 bidding 1.5 and bidder 2 bidding 1 is yet another Nash equilibrium.

As an optional aside note that at some level the above analysis applies to the ascending auction (also known as English auction) we run in class. Recall that in an ascending auction price starts at zero, and rises. At each moment, buyers indicate their willingness to continue bidding or decide to exit. The auction ends when just one bidder remains (if the last few bidders exit at the exact same moment, we need a tie-breaking rule as in the second-price sealed bid auction). The last bidder remaining in the auction wins the good, and pays the price at which the second remaining bidder dropped
out. The ascending auction is however more difficult to properly analyze than the second-price sealed-bid auction because when deciding when to exit bidders can condition their decision one when other bidders’ exited. In other words, bidders’ strategies can be much more complex than in the second-price sealed-bid auction. We will thus abstain from a careful analysis of the ascending auction.

2.2 First-Price Sealed-Bid Auction

The auction proceeds as follows:

- Bidders simultaneously submit bids.
- Bidder who submitted highest bid wins (if two or more bidders submitted the same bid which is higher than other bids, then we use a tie-breaking rule; say we allocate the units at random to bidders who submitted the highest bid and that each of these bidders has the same probability of getting the object).
- The winning bidder pays his own bid.

Recall the example in which there are two bidders. Bidder 1’s value is 1 and bidder 2’s value is 1. Is bidding one’s true value a dominant strategy? The first price-auction is not strategy-proof. For instance, if in Example 1 bidder 1 bids 0, then bidder 2 is better off bidding .5 than bidding 1.

**Exercise 2.** What are Nash equilibria of the first-price auction in the environment of Example 1?

In this exercise, the unique equilibrium of the first-price auction is both bidders bidding $b_1 = b_2 = 1$. Indeed, if one of the bidders bids more than 1 than this bidder wins and obtains negative payoff. If a bidder bids less than 1 then this bidder looses the auction and obtains payoff zero, the same he or
she would obtain by bidding 1. Hence bidding 1 is optimal when the other bidder also bids 1.

The reason this is the unique equilibrium resembles the standard Bertrand competition argument from Econ 101. First notice that no bidder wants to win while bidding more than 1. Thus, in any equilibrium each bidder bids at most 1 (if I bid more than 1 than I either win with positive probability and then lose money, or the other bidder bids even more, and then this other bidder is loosing money). At the same time no profile of strategies in which one of the bidders, say bidder 1, bids \( b_1 < 1 \) can be in an equilibrium, as then the other bidder would like to bid \( b_2 \) just above the first bidder and for every bid \( b_2 > b_1 \) any other bid in \((b_1, b_2)\) is a strict improvement for bidder 2.

As an optional aside notice that this auction resembles the descending price auction (also known as Dutch auction) we run in class. In a descending auction price starts high and drops continuously. At any point in time, a bidder can claim the good at the current price and pay that price (in case of ties, we again use the same tie-breaking rule as above). Auction ends as soon as some bidder claims the good. The two auctions are strategically equivalent: in both the strategy is just a single number (the bid, or at what price to stand up). When bidders bid \( b_i \) in the first-price sealed-bid auction they get exact same chances of winning, and pay the same amount, as if they planned to stand up at prices \( t_i = b_i \) in the descending auction. Notice that in the descending auction the price at which to stand up cannot be conditioned on other bidders’ behavior (just as the bid in the sealed bid auctions cannot be conditioned on other bidders’ bids). In this sense, the resemblance between the first-price sealed bid auction and the descending auction is closer than between the second-price sealed-bid auction and the ascending auction.
3 Auctions in Symmetric Incomplete Information Environment

We now study an environment in which from the perspective of a bidder, the values of the other bidders are random. In this more complex situation, each bidder would like to maximize his or her expected profits. If bidder $i$ with value $v_i$ has probability $\pi$ of winning the good and makes payment $t$ when he or she wins (and payment 0 otherwise) then the buyer’s expected profit is $\pi(v_i - t)$.

Another phrase for expected profits is average profits. The analysis of the case where bidders view the bids of others as random relies on concepts from probability (distribution, expected value) which you met in Econ 101.

Suppose there are $n$ bidders and each one of them draws values from two possible values $L$ and $H$ where $0 \leq L < H$. Let us denote by $\pi$ the probability a bidder has value $H$ and let us assume that knowing values of other bidders does not change the probability we assign to bidder $i$ having the high value. This assumption is known as independence (values are said to be independently distributed) and the model is known as the model of independent private values. In particular, from perspective of bidder 1 with value $L$ bidder 2 might have value $L$ with probability $1 - \pi$ and value $H$ with probability $\pi$; if bidder 1 had value $H$, he or she would still assign probability $\pi$ to bidder 2 having value $H$.

**Exercise 3.** Suppose there are three bidders 1, 2, and 3, and that bidder 1 has value $L$. What probability does this bidder assign to the possibility (also known as event) that bidder 2 has value $L$ and bidder 3 has value $H$? The independence means that this probability is the product of the probability $(1 - \pi)$ that bidder 2 has value $L$ and the probability $\pi$ that bidder 3 has value $H$.

These probabilities are important in understanding bidders payoffs. A

\[2\text{Notice that if we define } T = \pi t \text{ to be the expected payment, then we can equivalently write the expected payoff as } \pi v_i - T.\]
bidder with value $H$ who wins the object and pays $t$ has payoff $H - t$. However the probability of winning depends on other bidders’ bids, and in the second-price auction also the payment of the winning bidder depends on other bidders’ bids. We now assume that bidders want to maximize their expected payoffs, that is payoffs across all possible events weighted by probabilities of those events.

Despite the presence of private information, the concept of strategy-proofness is essentially unchanged: a mechanism is strategy proof if bidding one’s value is a dominant strategy, that is a bidder wants to do it irrespective of the bids of others; in other words, bidding one’s true value maximizes the expected payoff of a bidder.

Also the concept of Nash equilibrium is essentially the same except that now each bidder’s strategy specifies how this bidder bids having low value and how this bidder bids having high value. Let us denote the bid of bidder $i$ with low value $b_i^L$ and the bid of bidder $i$ with high value $b_i^H$. We refer to this pair of bids as a strategy of bidder $i$. Thus a bidder who submits some bid $b$ might not know whether he or she wins even if there are no ties; even without a tie the bidders now maximize their expected payoffs. To underline these differences with Nash equilibrium we often refer to the equilibrium as the Bayesian-Nash equilibrium; other times we refer to this concept simply as equilibrium (just as we often refer to Nash equilibrium simply as equilibrium).

In equilibrium, a bidder knows the distribution of values of other bidders (that is the bidders knows that the values are independent, take values $L$ and $H$ and the probability of the high value is $\pi$); the bidder also knows the realization of his or her own value but the bidder does not know the realizations of the values of other bidders. While no bidder knows what value the other bidder has, we assume that each bidder correctly understands that another bidder, say bidder $j$, bids $b_j^L$ when having low value and $b_j^H$ when having high value.

**Definition 1.** A set of bidding strategies is a Bayesian-Nash equilibrium if
each bidder’s strategy maximizes his expected payoff given the strategies of
the others.

As an aside let me say that we will be only interested in so-called “pure-
strategy” Bayesian-Nash equilibria, that is we will be assuming that each
bidder’s private information (in this case, his or her value $v_i$) uniquely deter-
mines the bidder’s bid, just as in the description above.

3.1 Second-price Auctions

Suppose that all bids are allowed $G = [0, +\infty)$. Is second-price auction still
strategy-proof? Yes, the same argument as before establishes it.

What are Bayesian Nash equilibria? Bidding truthfully $b_i = v_i$ is an
equilibrium (notice that $b_i = v_i$ means that $b_i^L = L$ and $b_i^H = H$). There are
other equilibria. For instance, if $v_i \in \{0, 1\}$ and takes the high value with
probability $\pi = \frac{1}{3}$ then bidding $b_1 = \frac{1}{2} v_1$ and $b_2 = v_2$ forms an equilibrium.

3.2 First-price Auctions

Suppose that all bids are allowed $G = [0, +\infty)$. Is second-price auction still
strategy-proof? What are (pure-strategy) Bayesian Nash equilibria?

It turns out that when all nonnegative bids are allowed and bidders have
private information, then there is no (pure-strategy) Bayesian Nash equilib-
rium. Suppose for instance that $L = 0$ and $H = 3$, and $\pi = \frac{1}{2}$. Is the profile
of strategies $b_i = \frac{1}{3} v_i$ an equilibrium? No, it is not because when bidders bid
this profile of strategies then the expected payoff of bidder 1 with value $H$
is $\frac{1}{2} (3 - 1) + \frac{1}{2} \frac{1}{2} (3 - 1) = 1.5$; notice that here the first $\frac{1}{2}$ is the probability
bidder 1 faces bidder 2 with low value (and then wins), and that the mul-
tiplication $\frac{1}{2} \frac{1}{2}$ is the probability that bidder faces bidder 2 with high value
(probability $\frac{1}{2}$) and gets the object in tie-breaking (conditional probability
of $\frac{1}{2}$). Bidder 1 with high value can achieve better expected payoff than 1.5
by bidding 1.01. Indeed, then this bidder always wins his or her expected
payoff is $3 - 1.01 = 1.99 > 1.5$. 

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4 Incomplete Information and Restricted Bid Spaces

In Lectures 15 and 16 we will study the same model as in Section 3 except that we will assume that only two bid levels are allowed $\ell$ and $h$ where $0 \leq \ell \leq h$.\(^3\) If $\ell > L$ we will be assuming that bidders with value $L$ do not participate in the auction. This is related to individual rationality we discussed in the school choice context.

**Exercise 4.** Suppose there are two bidders, the two possible values are $L = 0$ and $H = 1$, and the probability of the high value $v_1 = 1$ is $\pi = \frac{1}{2}$.

- Suppose that only two bids $\ell = 0$ and $h = 1$ are allowed. What is an equilibrium of the second-price auction? What is an equilibrium of the first-price auction?

- Suppose only two bids $\ell = 0$ and $h = \frac{1}{2}$ are allowed. What is an equilibrium of the second-price auction? What is an equilibrium of the first-price auction?

\(^3\)I would like to thank John Riley for suggesting the restriction of the bid space as a way to ensure the existence of pure-strategy equilibria in first-price auctions.