Notes for Lecture 15 and 16
Auction Theory (Part 2 and 3)

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This note: (1) discusses efficiency in auctions; (2) analyzes equilibria of the second-price and first-price auctions with limited bid spaces; (3) constructs revenue-maximizing bid spaces for sellers who always want to sell; and (4) analyzes reserve prices and their impact on revenue.

We continue to consider the bidding environment introduced in Lecture 14 (so called, independent private values environment/model).

1 Efficiency

An outcome of an auction is Pareto efficient (or, simply, efficient) if no other outcome is weakly better for every parties involved (the seller and all bidders) and strictly better for at least one party. This is the standard definition of Pareto efficiency we worked with in Parts 1 and 2 of this course. Notice that here it is equivalent to assigning the object to the party with highest value.

**Theorem 1.** An outcome of an auction is Pareto efficient if and only if the party (seller or bidder) with the highest value has the object.

*Why is this theorem true?* Suppose an agent (bidder or seller) $L$ has the object and another agent $H$ has strictly higher value for the object: then
there is a Pareto improvement in which we give the object to agent $H$, and have him or her compensate in money agent $L$, hence the original outcome is Pareto efficient. Suppose now that the highest value agent $i$ has the object:\footnote{Notice that this agent might have high or low value; all the matters is that none else has higher value.} by just moving money around we cannot create a Pareto improvement as at the end of such a distribution all agents would need to have at least as much money as they initially had. Giving the object to another agent $j$ does not help either as agent $i$ would need to be compensated and the only agent who is willing to pay is agent $j$ who receives the object. But since agent $j$ has weakly lower value for the object than agent $i$, $j$’s cannot gain by receiving the object and losing the monetary equivalent of $i$’s value of the object. (As usual, this and other proofs are optional).

Notice that when two parties have the highest value, assigning the good to either of them is equally efficient. Let us globally assume that the value of the seller is zero while the values of the bidders are nonnegative. This allows us to conclude that whenever one of the bidders receives the good being sold, then for the efficiency of such an outcome it is both sufficient and necessary that this bidder’s value is weakly higher than all other bidders’ values.

Recall the equilibria we analyzed in Lecture 14. Are the equilibria of the first-price auction efficient? Are the equilibria of the second-price auction efficient?

Furthermore, we say that an auction is Pareto efficient if it generates Pareto efficient outcomes irrespective of the realization of bidders’ values. For instance, if each bidder can have value $L$ or $H$ (where $H > L > 0$), then the auction is efficient if whenever at least of the bidders drew value $H$ then the auctions allocates the object to one of the bidders with value $H$.\footnote{Notice that this agent might have high or low value; all the matters is that none else has higher value.}
2 Expected Revenues and Payoffs

Suppose there are two bidders, the two possible values are \( L = 0 \) and \( H = 1 \), and the probability of the high value \( v_1 = 1 \) is \( \pi = \frac{1}{3} \). What is the seller’s revenue in the “bidding one’s value” equilibrium of the second-price auction?

The second highest bid is non-zero only when both bidders have high value; the second highest bid is then 1 and this happens with probability \( \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \). Hence the expected revenue is \( \frac{1}{9} \cdot 1 = \frac{1}{9} \).

In class we calculated bidder’s payoffs in many examples. In the example above, the expected payoff of a bidder with value 0 is 0. The expected payoff of a bidder with value 1 is

\[
\frac{2}{3} (1 - 0) + \frac{1}{3} \frac{1}{2} (1 - 1) = \frac{2}{3}.
\]

3 Incomplete Information and Restricted Bid Spaces

Let us now restrict the space of allowed bids in the model of Section 3 of Lecture 14. We assume that only two bid levels are allowed \( \ell \) and \( h \) where \( 0 \leq \ell \leq h \). If \( \ell > L \) we will be assuming that bidders with value \( L \) do not participate in the auction. This is related to individual rationality we discussed in the school choice context.

As a preparation, notice that in both first-price and second-price auctions, a bidder with value \( v \in \{L, H\} \) maximizes \( pv - T \) where \( p \) is the probability of winning and \( T \) is the expected payment. In the first price auction \( T \) equals bidder’s bid, in the second-price auction \( T \) is the expected value of the highest bid submitted by \( n - 1 \) other bidders.

**Example 1.** Suppose there are two bidders, the two possible values are \( L = 0 \) and \( H = 1 \), and the probability of the high value \( v_1 = 1 \) is \( \pi = \frac{1}{3} \).

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\( ^2 \)I would like to thank John Riley for suggesting the restriction of the bid space as a way to ensure the existence of pure-strategy equilibria in first-price auctions.
Suppose that only two bids $\ell = 0$ and $h = 1$ are allowed.

- Find an equilibrium of the second-price auction. Is the equilibrium efficient? What is the expected revenue of the seller?

Consider the second-price auction. Since the bids allow the bidders to bid truthfully, strategy-proofness establishes that truthful bidding is an equilibrium in the second price auction. Are there other equilibria? Yes, for instance bidder 1 bidding 1 irrespective of his or her value and bidder 2 bidding 0 irrespective of his or her value is also an equilibrium. Notice that this last equilibrium is not efficient when bidder 1 has low value and bidder 2 has high value. In the sequel we will focus attention on symmetric equilibria that is on the equilibria in which both bidders bid the same $b_L$ when having low value and they both bid the same $b_H$ when they have high value.

- Find an equilibrium of the first-price auction. Is the equilibrium efficient? What is the expected revenue of the seller?

In the first price auction, does it make sense for bidder with value $v = 0$ submit a bid $b = 1$? No, as the expected payoff is then 0 while the bidder could bid 0 and obtain strictly positive expected payoff. At the same time bidding 0 gives nonnegative profits to both bidders, and thus it is an equilibrium for all bidders to bid 0 regardless of their value. Such an equilibrium is not efficient as in the resulting tie-breaking sometimes a low-value bidder obtains the good even when the other bidder has high value.

**Example 2.** Consider the value structure of Example 1 and suppose that only two bids $\ell = 0$ and $h = \frac{1}{2}$ are allowed.

- *Find an equilibrium of the second-price auction. Is the equilibrium efficient? What is the expected revenue of the seller?*

In the auction with bids in $(0, \frac{1}{2})$ bidders with value $H$ cannot bid truthfully. Still we can check that a profile of strategies in which each low-value bidder bids 0 and each high-value bidder bids $\frac{1}{2}$ is an equilibrium.
Indeed, strategy-proofness still implies that bidder with value 0 cannot do better than to bid 0. In fact, bidding $\frac{1}{2}$ is actually strictly worse than bidding 0 for low value bidder: indeed, by submitting this high bid the low value bidder would either lose the auction (and have payoff 0), or win the auction and pay 0 (and have payoff 0), or win the auction and pay $\frac{1}{2}$ (and have strictly negative payoff). In symmetric equilibrium in which low value bidders bid $\frac{1}{2}$ the last of this outcomes would have strictly positive probability, and hence the expected payoff of low value bidder from bidding $\frac{1}{2}$ would be strictly negative, while this bidder can achieve payoff of zero by bidding 0 or not participating in the auction.

It remains to check the incentives of bidders with high value (1). By bidding $\frac{1}{2}$ such bidder obtains the expected payoff of $\frac{2}{3} (1 - 0) + \frac{1}{3} \left(1 - \frac{1}{2}\right) = \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$. By deviating from the equilibrium prescriptions and bidding 0, the high-value bidder obtains the expected payoff of $\frac{2}{3} (1 - 0) = \frac{1}{3}$. Hence bidding $\frac{1}{2}$ is better for high-value bidders.

As an aside let us check whether bidders with value 1 might prefer to bid 0 over bidding $\frac{1}{2}$ when the other high-value bidder bids 0? No, as then bidding $\frac{1}{2}$ would not change the payment but it would raise the probability of winning from $\frac{1}{2}$ to 1 (and winning and paying 0 has strictly positive payoff for these bidders).

The equilibrium is efficient as the high-value bidders bid more than low-value bidders thus guaranteeing that the highest value bidder obtains the good. The expected revenue of the seller is $\frac{1}{3} \frac{1}{3} \frac{1}{2} = \frac{1}{18}$ and it is lower than the expected revenue in equilibrium with bids $\{0, 1\}$.

- *Find an equilibrium of the first-price auction. Is the equilibrium efficient? What is the expected revenue of the seller?*

Recall that only two bids $\ell = 0$ and $h = \frac{1}{2}$ are allowed. Let us check whether it is an equilibrium for bidders with low value to submit the low bid, and for bidders with high value to submit the high bid.

First consider a bidder who has low value. By bidding 0 this bidder obtains the expected payoff of 0. This is as good as not participating in
the auction and better than bidding \( \frac{1}{2} \). Indeed, bidding \( \frac{1}{2} \) leads to negative expected payoff as the bidder has positive probability of winning the auction and paying \( \frac{1}{2} \) for an object worth 0.

Second consider a bidder with high value. For this bidder any bid has strictly positive payoff; in particular, any bid is better than not participating in the auction. The expected payoff of bidding \( \frac{1}{2} \) is \( \frac{2}{3} \left( 1 - \frac{1}{2} \right) + \frac{1}{3} \left( 1 - \frac{1}{2} \right) = \left( \frac{2}{3} + \frac{1}{3} \right) \left( 1 - \frac{1}{2} \right) = \frac{5}{6} = \frac{5}{12} \). The expected payoff of bidding 0 is \( \frac{2}{3} \left( 1 - 0 \right) = \frac{2}{3} \). Hence any bidder with high value prefers to submit the high bid.

We can conclude that each bidder weakly prefers to follow the strategy from the profile of strategies we analyzed given that other bidders do so; hence the profile we analyzed is indeed an equilibrium.

The resulting auction is efficient as bidders with high value bid more than bidders with low values, and thus the auction is always won by a highest-value bidder.

To calculate the seller’s expected revenue notice that the seller collects the payment of 0 if both bidders have low value, and the payment of \( \frac{1}{2} \) if at least one bidder has high value. Thus the the expected revenue equals

\[
\left( \frac{2}{3} \right) \left( \frac{2}{3} \right) \cdot 0 + \left( 1 - \left( \frac{2}{3} \right) \left( \frac{2}{3} \right) \right) \frac{1}{2} = \left( \frac{5}{9} \right) \frac{1}{2} = \frac{5}{18}.
\]

4 Designing the Auctions

As in Example 1, suppose there are two bidders, the two possible values are \( L = 0 \) and \( H = 1 \), and the probability of the high value \( v_1 = 1 \) is \( \pi = \frac{1}{3} \). We will now consider the design problem of the seller.

4.1 Second-price Auction

Suppose the seller uses a second-price auction and wants to sell the object irrespective of the types of bidders. In particular, this implies that the seller needs to offer a bid level that is weakly below the low value \( L \) (otherwise the low value bidder would prefer not to participate in the auction). We want
to help the seller design such auction in a way that maximizes the seller’s revenue in an equilibrium. We restrict attention to symmetric equilibria: all bidders with the same value are supposed to bid the same.³

Notice that it is sufficient to allow only two bid levels $\ell$ and $h$, where $\ell$ is chosen by low value bidders and $h$ is chosen by high value bidders. We have already noted that $\ell$ is weakly lower than $L$? Does it make sense to set $\ell < L$?

**Theorem 2.** Setting $\ell = L$ and then optimally choosing $h$ brings strictly higher revenue then setting $\ell < L$ and then optimally choosing $h$.

This theorem is true in general and not only in the example. It might be useful in answering problem sets and exams. As usual, the proof is optional. I will develop a proof sketched without details in Lecture 1 (in Lecture 2 we have used an approach which relied on the numerical values of the example we discussed).

First notice that $\ell \leq L$ and optimally chosen $h$ guarantee that the allocation is efficient. Indeed, the seller will chose $h > \ell$ and such that the high-value bidder will bid $h$. Since $\ell \leq L$ both bidders participate irrespectively of the values they drawn, and the high value bidder bids more, guaranteeing efficiency of the resulting allocation.

Now consider bidders’ payoffs. By setting $\ell = L$ the seller gives 0 expected payoff to low-value bidders; by setting $\ell < L$ the seller gives strictly positive expected payoffs to these bidders. Furthermore, the optimally chosen $h$ is the highest $h$ such that the high value bidder weakly prefers bidding $h$ over bidding $\ell$. In effect, at optimally chosen $h$ the high-value bidder is actually indifferent between bidding $h$ and $\ell$, and such a bidder expected payoff thus equals $(1 - \pi) \frac{1}{2} (H - \ell)$ (that is the payoff such a bidder would obtain bidding $\ell$ while the other bidder bids $\ell$ having low value and $h$ having high value). In effect, the high value bidder’s expected payoff is higher for $\ell < L$ than for $\ell = L$.

³We assume throughout that all bidders have access to the same bid set $G$.  

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To conclude the proof let us define the total surplus as the sum of the expected payoffs of the seller and all bidders.\(^4\) Thus the seller’s revenue equals to the total surplus minus expected payoffs of all bidders. As long as we assign the good in an efficient way, the seller’s surplus increases if he or she gives lower expected payoffs to the bidders. As we have shown above setting \(\ell = L\) gives each bidder lower expected payoff than setting \(\ell < L\) whether this bidder has low or high value. Hence, \(\ell = L\) must give the seller higher revenue than \(\ell < L\).

Given Theorem 2, we may assume that \(\ell = L\). What is the optimal value of \(h\) that the seller can set? The higher \(h\), the higher seller’s revenue as long as the high-value bidder still chooses to bid \(h\). In our example, this means that the expected payoff of a high-value bidder bidding \(h\) must be weakly above the expected payoff of such a bidder bidding \(\ell = L\). The former equals
\[
\frac{2}{3} (1 - 0) + \frac{1}{3} \frac{1}{2} (1 - h)
\]
while the latter equals
\[
\frac{2}{3} \frac{1}{2} (1 - 0).
\]
Since the seller’s revenue is increasing in \(h\) as long as \(h\) satisfies the above constraint, the seller optimally sets \(h\) such that the two expected payoffs above are equal:
\[
\frac{2}{3} (1 - 0) + \frac{1}{3} \frac{1}{2} (1 - h) = \frac{2}{3} \frac{1}{2} (1 - 0).
\]
This allows us to calculate the optimal \(h\) as
\[
h = 3.
\]

Is the resulting auction efficient? Yes, all bidders participate and high-value bidders bid more than low-value bidders, thus assuring that the object

\(^4\)Notice that the total surplus of an efficient allocation is \((1 - \pi)^2 L + 2\pi (1 - \pi) H + \pi^2 H\).
is allocated to the highest-value bidder.

What is the resulting expected revenue? The seller gets 0 unless both bidders have high value. The expected revenue is thus

\[ \frac{1}{3} h = \frac{1}{3}. \]

4.2 First-price Auction

Suppose now that the seller uses a first-price auction and wants to sell the object irrespective of the types of bidders. As before, this implies that the seller needs to offer a bid level that is weakly below the low value \( L \) (otherwise the low value bidder would prefer not to participate in the auction). Also, as before, we want to help the seller design such auction in a way that maximizes the seller’s revenue in an equilibrium and we restrict attention to symmetric equilibria: all bidders with the same value are supposed to bid the same.\(^5\)

Notice that it is sufficient to allow only two bid levels \( \ell \) and \( h \), where \( \ell \) is chosen by low value bidders and \( h \) is chosen by high value bidders. As in the case of second-price auctions, setting \( \ell = L \) yields strictly higher revenue than setting any \( \ell < L \) provided \( h \) is set optimally in both cases.

**Theorem 3.** Setting \( \ell = L \) and then optimally choosing \( h \) brings strictly higher revenue then setting \( \ell < L \) and then optimally choosing \( h \).

The proof of this result follows the exact same steps as the proof of the analogous result for second-price auctions (the proof is of course optional).

In the sequel we assume that \( \ell = L \). **What is the optimal value of \( h \) that the seller can set? Is the resulting auction efficient? What is the resulting expected revenue?**

As in the analysis of the design of second-price auctions, we will focus on symmetric equilibria, that is we will assume that all bidders have the same strategy of mapping their values to their bids, that is they bid the same \( b_L \) when they have low value and they bid the same \( b_H \) when they have high value.

\(^5\)We assume throughout that all bidders have access to the same bid set \( G \).
value. We have seen in class that setting $h$ too high leads to the collapse of the equilibrium in which high-value bidders bid $h$. Indeed, setting $h \geq H$ implies that every high-value bidder bidding $h$ gets weakly negative payoff, while by bidding $\ell = L$ such a bidder gets a strictly positive payoff. At the same time as long as high value bidders are willing to bid $h$ the seller would like to set $h$ as high as possible as in line with the calculation above in our analysis of Example 2 seller’s revenue equals $(1 - \frac{2}{3})h = \frac{5}{9}h$ in every equilibrium in which low value bidders bid 0 and high value bidders bid $h$. Our analysis of Example 2 also shows that bidding $h$ gives the high-value bidder expected payoff of $\left(\frac{2}{3} + \frac{1}{3}h\right)(1 - h) = \frac{5}{6}(1 - h)$ and bidding 0 gives the high-value bidder expected payoff of $\left(\frac{2}{3}\right)(1 - 0) = \frac{2}{6}$. The high value bidder is thus willing to bid $h$ as long as

$$\frac{5}{6}(1 - h) \geq \frac{2}{6}$$

and the highest bid $h$ satisfying this inequality is $h = 1 - \frac{2}{6} = 1 - \frac{2}{5} = \frac{3}{5}$. This highest bid value yields the highest revenue and the resulting revenue is $\frac{5}{9}h = \frac{5}{9} \cdot \frac{3}{5} = \frac{3}{9} = \frac{1}{3}$.

### 4.3 Revenue comparison

Which auction raises more revenue? First-price auction or second-price auction? Remarkably in our Example 2 both auctions with properly designed bids yield the exact same revenue. This is a general regularity, first established by Vickrey (1961) in a slightly different model.\(^6\)

**Theorem 4.** Suppose that first-price and second-price auctions both have efficient equilibria and are designed in a way that raises the highest expected revenue among such equilibria. Then, the seller’s expected revenue is the same in both auctions.

The reason this result is true resembles our analysis of why we want to

\(^6\)The form of revenue equivalence in our notes is new.
set \( \ell = L \). Seller’s expected revenue is equal to total surplus minus the sum of bidders’ expected payoff. Efficiency implies that the total surplus is the same in both auctions. Furthermore, in an optimally designed auction a bidder with low value has zero surplus (as we set \( \ell = L \)) and the bidder with high value is indifferent between low and high bid. Since the expected payoff of high-value bidder submitting low bid is the same in both auctions, the revenue equivalence follows (the proof, as usual, is optional).

5 Reserve Prices

In Section 4 we maximized seller’s revenue assuming that the seller wants to sell the object irrespective of bidders’ values. But is it always optimal? Or can it be that the seller sometimes prefers to sell only to bidders with high values? Selling to only bidders with value above some threshold is known as setting the reserve price (see the last paragraph of this section for how such reserve prices modify the auction formats we study).

In Example 1, selling only to high value bidder means that we can set a fixed price of \( h = 1 \). Bidders with high value will be willing to pay this price while they are not willing to pay any higher price. Hence if we sell only to high value bidders than \( h = \ell = 1 \) is optimal and the resulting revenue is 1 with probability there is at least one high value bidder; this is the same probability as 1 minus the probability there are no high value bidders, that is the probability is \( 1 - \left( \frac{2}{3} \right)^2 = \frac{5}{9} \). Thus, the expected revenue when selling only to high-value bidders at price 1 equals \( \frac{5}{9} \times 1 = \frac{5}{9} \). This revenue is higher than the optimal revenue we calculated in Section 4. this however is not necessarily the case: if \( L \) is high then it might be that the seller’s expected revenue is higher in the type of second-price auction we run in Section 4 (that caters to both high- and low-value bidder) than in an auction that only targets high-value bidders.

In any environment with just two value types, the reserve prices effectively mean selling only to the highest value bidders, and then it is optimal to set
the reserve price to be equal to their value. We thus obtain:

**Theorem 5.** In both the first-price and the second-price auctions, to maximize revenue either (i) we sell to both low- and high-value bidders, or (ii) we sell only to high-value bidders at price $H$ (we then refer to this price as optimal reserve price excluding low-value bidders).

As an optional aside (not required on exams) let me comment on the fact that the reserve price idea is more general than this simple implementation in our two-value model. In general, a second price auction with reserve price $R$ works as follows: each participating bidder submits a bid; to win the bidder needs to bid above the reserve price, if there are bidders who bid above the reserve price, the highest of such bidders wins, and pays $\max(R, \text{the second highest bid})$ (this max equals $R$ when there are no bids other than the winning bid). A first-price auction with reserve price $R$ works as follows: each participating bidder submits a bid; to win the bidder needs to bid above the reserve price, if there are bidders who bid above the reserve price, the highest of such bidders wins, and pays his or her bid.