4 parts, 100 points total. Please answer questions and provide arguments when asked for. Please make sure to connect the argument to the problem at hand – stating the definition does not count as an argument. For instance, if you are asked to show that a matching is stable, it is not enough to say that it is stable because there are no blocking pairs and no blocking individuals. You may rely on all results from lecture notes, no need to re-prove them (this only applies to results from the notes; results not from the notes would need to be proven).

**Part 1: Two-sided Matching (24 points; each question worth 6 points)** Consider an economy with 4 graduating medical students (Adrien, Bill, Claire, and Deborah) and 4 hospitals ($h_1, h_2, h_3,$ and $h_4$). Each student wants to be matched with at most one hospital, and each hospital wants to be matched with at most one student. The preferences of the students and hospitals, respectively, are:

<table>
<thead>
<tr>
<th>Adrien</th>
<th>Bill</th>
<th>Claire</th>
<th>Deborah</th>
</tr>
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<tr>
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<td>Adrien</td>
<td>Claire</td>
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<tr>
<td>Claire</td>
<td>Bill</td>
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<td>Bill</td>
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<tr>
<td>Adrien</td>
<td>Claire</td>
<td>Adrien</td>
<td>Claire</td>
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</tbody>
</table>

1. **What is the outcome of the DA with students proposing? Please describe the run of the algorithm.**

   **Round 1:**
   
   $A \rightarrow 2$
   $B \rightarrow 4$
   $C \rightarrow 1$
   $D \rightarrow 2$ (rejected)

   **Round 2:**
   
   $A \rightarrow 2$
   $B \rightarrow 4$ (rejected)
   $C \rightarrow 1$
   $D \rightarrow 4$
Round 3:

\[
\begin{align*}
A & \rightarrow 2 \\
B & \rightarrow 3 \\
C & \rightarrow 1 \\
D & \rightarrow 4
\end{align*}
\]

Doctor-proposing DA matching: \{(A, 2), (B, 3), (C, 1), (D, 4)\}

2. Is there a stable matching in which \( h_3 \) and Claire are matched with one another? If yes, give an example of such a matching; if no, provide an argument why not.

Suppose there is such a matching and let \( x, y, \) and \( z \) be the hospitals other than \( h_3 \). Then the matching takes the following form:

\[
\{(A, x), (B, y), (C, 3), (D, z)\}
\]

For stability, we need to rule out any blocking pairs. \( C \) would prefer 2 and 1 over 3, so we need to make sure that both 2 and 1 are with doctors that they prefer to 1. This means that \((D, 1)\) must be a pair, and therefore \( z = 1 \). However, notice that however \( h_2 \) and \( h_4 \) are allocated between \( A \) and \( B \), \((D, 4)\) will always be a blocking pair, since \( D \) prefers 4 to 1 and 4 prefers \( D \) to all other hospitals. This proves that there is no stable allocation with \((C, 3)\) as a pair.

3. Assume that agents report preference rankings and then the student-proposing DA is run to determine who matches with whom. Assume also that all agents report their true preferences except possibly hospital \( h_1 \). Can hospital \( h_1 \) improve their matching outcome by reporting a preference ranking different from their true ranking? If yes, please provide (i) a preference ranking such that if \( h_1 \) reports it then they get a better match than they would by reporting their true ranking, and (ii) please provide the resulting matching. If no, please give an argument why not.

Recall that the hospital-proposing DA matching is \{(1, C), (2, A), (3, B), (4, D)\}

Since \( h_1 \) is paired with \( C \) in the hospital-proposing DA matching, this means that they cannot obtain a better stable matching, therefore, there does not exist a preference ranking they can report that improves their matching.

4. Is the matching \{(h_1, Claire), (h_2, Bill), (h_3, Adrien), (h_4, Deborah)\} stable? Why or why not?

This matching is not stable as \((h_2, A)\) and \((h_3, B)\) are blocking pairs:
Part 2: Object Allocation without Transfers (24 points; each question worth 6 points)
Consider an economy with 4 students $s_1, \ldots, s_4$ and 4 schools $A, B, C, D$.

The preferences of the students and the priorities of schools, respectively, are

$s_1: B > C > A > D$ \quad A: s_1 > s_2 > s_3 > s_4$
$s_2: A > B > C > D$ \quad B: s_2 > s_3 > s_1 > s_4$
$s_3: B > A > D > C$ \quad C: s_3 > s_1 > s_4 > s_2$
$s_4: D > A > B > C$ \quad D: s_4 > s_3 > s_1 > s_2$

1. Consider the allocation $\{(s_1, C), (s_2, A), (s_3, B), (s_4, D)\}$. Is it efficient? Why or why not?

An allocation is efficient if we cannot change it in such a way that makes one student better off while keeping all others equally as well off. Since $s_2$ is with their most preferred school, they must be paired with $A$ in any new allocation. The same is true for $s_3$ and $s_4$. But if we cannot change any of these three pairs, it is not possible to construct a new matching. Therefore, the existing one is stable.

2. For this question only, assume that the school seats are allocated to students via a Top-Trading-Cycles mechanism in which student $s_1$ controls seat in school $A$, student $s_2$ controls seat in school $B$, student $s_3$ controls $C$, and student $s_4$ controls $D$. What is the resulting allocation? Please describe the run of the algorithm.

Round 1:
$s_1 \rightarrow B \rightarrow s_2 \rightarrow A \rightarrow s_1$,
$s_2 \rightarrow A \rightarrow s_1 \ldots$,
$s_3 \rightarrow B \rightarrow s_2 \rightarrow A \rightarrow s_1 \ldots$,
$s_4 \rightarrow D \rightarrow s_4$.

The outcome of round 1 are the following pairs: $(s_1, B); (s_2, A); (s_4, D)$

Round 2:
$s_3 \rightarrow C \rightarrow s_3$

The outcome of round 3 is the pair $(s_3, C)$

The resulting Top-Trading-Cycles matching is $\{(s_1, B), (s_2, A), (s_3, C), (s_4, D)\}$

3. For this question assume that the school seats are allocated via DA. What is the allocation generated by the DA with students proposing? Is there another stable allocation that Pareto dominates (for students) this allocation? In other words, is there a stable allocation of school seats to students in which no student is worse off and some students are better off? If yes, please write down the Pareto-dominant stable allocation; if not, please provide an argument why not.
First, we run student-proposing DA:

Round 1:

1 → B (rejected)
2 → A
3 → B
4 → D

Round 2:

1 → C
2 → A
3 → B
4 → D

The student-proposing DA matching is \{(s_1, C), (s_2, A), (s_3, B), (s_4, D)\}

This matching is Pareto efficient, as we already verified in Question 1, and hence there is no other matching that is a Pareto improvement.

4. Assume that the seats are allocated in the Boston Mechanism and that all students but possibly s_3 submit their true preference rankings to the mechanism. Can s_1 improve her allocation by submitting a ranking different from her true preference ranking? If yes, please provide an improving ranking; if not, please provide an argument why not.

First, we run the Boston mechanism with the original set of preferences:

Round 1:

1 → B
2 → A
3 → B
4 → D

The following final allocations are made: \{(s_2, A), (s_3, B), (s_4, D)\}

Round 2:

1 → C
The Boston mechanism matching is \( \{(s_1, C), (s_2, A), (s_3, B), (s_4, D)\} \)

Since \( s_3 \) is matched with their most preferred school, they cannot improve their matching by misreporting their preferences.

**Part 3: Auctions 36 points; each question worth 6 points** A monopolistic seller has a single indivisible good for sale. There are five potential buyers (we will also call them bidders). Each buyer \( i = 1, 2, 3, 4, 5 \) has value \( v_i \) for the auctioned good. Each \( v_i \) is an independent draw from the uniform distribution on the interval \([4, 12]\).

For questions 1 and 2, assume that the good is sold in a sealed-bid second price auction.

1. **Find an equilibrium and provide the argument (that is check that the bidders best-respond to each other).**

The equilibrium in a second price auction is for each bidder to bid their true valuation of the good. We know from the lecture notes that this is a dominant strategy.

2. **What is the expected revenue of the seller? Please calculate it to two decimal points.**

Denote the expected revenue of the seller by \( E[\Pi] \); the expected revenue equals the expected amount paid for the item. If all players play according to the above strategy, this is the expected second highest bid, which is the expected second highest value.

As in class, the expected 1st, 2nd, ..., 5th highest value partitions the interval \([4, 12]\) into six intervals of equal length. Thus, the expected second highest value is \( 12 - 2 \times \frac{8}{6} = \frac{28}{3} \), and this is the expected revenue of the seller.

For questions 3-6, assume that the good is sold in a sealed-bid first price auction.

3. **Find equilibrium bids and provide an argument that they are in equilibrium (that is check that the bidders best-respond to each other).**

We know from Theorem 1 in lecture notes 14 that the equilibrium bidding strategy in a first price, sealed-bid auction with 5 bidders whose values are independently and uniformly drawn from \([0, 1]\) is \( b(v) = \frac{4}{5} v \). By Lemma 1, the same strategies are in equilibrium when the values are drawn from \([0, 8]\). By Lemma 2, the equilibrium bid when values are drawn from \([4, 4+8]\) is:

\[
b = \frac{4}{5}(v - 4) + 4
\]
4. What is buyer 1’s expected profit from participating in the auction? Please (i) express it in terms of \( v_1 \) providing your calculation/argument, and (ii) compute the expected profit when \( v_1 = 8 \) (show your work).

Bidder 1’s expected profit is their winning probability multiplied by their utility when they win. Since the bidding function is increasing in a bidder’s value, the probability that any player wins is simply the probability that their value is the highest value. Thus the expected payoff to bidder 1 when their value is \( v \) equals

\[
E[\pi_1|v] = P(v > v_j \forall j \neq 1)(v - b(v)) = \left( \frac{v - 4}{5} \right)^4 \left( \frac{v - 4v + 4}{5} \right) = \left( \frac{v - 4}{8} \right)^4 \left( \frac{v - 4}{5} \right)
\]

When \( v = 8 \), the expected payoff is

\[
E[\pi_1|v = 8] = \left( \frac{8 - 4}{8} \right)^4 \left( \frac{8 - 4}{5} \right) = \left( \frac{1}{2} \right)^4 \left( \frac{4}{5} \right) = 0.05
\]

5. What is the revenue of the seller in the above equilibrium when the the bidders’ values are \( v_1 = 5, v_2 = 7, v_3 = 9, v_4 = 10, \) and \( v_5 = 10.5 \)? Please calculate the revenue, up to two decimal points.

The bidder with the highest value wins the auction and pays \( b = \frac{4}{5}(9.5 - 4) + 4 = \frac{42}{5} = 8.4 \), which is the seller’s revenue.

In all above questions we assumed that the bidders do not know each others’ values. For the last question, let us assume that the bidders know each others’ values, and that the values are \( v_1 = 9.5, v_2 = 9, v_3 = 8, v_4 = 6, \) and \( v_5 = 4 \), respectively.

6. Please find an equilibrium in which bidder 5 submits bid \( b_1 = 9 \). Provide an argument.

A possible equilibrium is the following: bidder 1 submits a bid of 9, and bidder 2 randomizes uniformly over bids between 8.5 and 9 (\( b_2 \sim U[8.5, 9] \)). The remaining bidders all bid 5.

Bidders 2, 3, 4, and 5 have no incentive to deviate since they would only be able to change the outcome of the auction by bidding above the bid of bidder 1, thereby earning a negative payoff. Now suppose bidder 1 were to deviate and bid slightly less than 9. More specifically, suppose they bid \( 9 - \epsilon \), where \( \epsilon > 0 \). Their expected payoff from submitting such a bid is the following:

\[
E[\pi_1|b = 9 - \epsilon] = P(9 - \epsilon > b_2) \times (9.5 - (9 - \epsilon)) = 2 \times (9 - \epsilon - 8.5) \times \left( \frac{1}{2} + \epsilon \right) = \frac{1}{2} - 2\epsilon^2,
\]

which is always less than \( \frac{1}{2} \), the amount that bidder 1 obtains when they bid 9. Since no bidder has an incentive to deviate, this strategy profile is an equilibrium.
Part 4: Reserve Prices (16 points; questions 1-2 worth 6 points, question 3 worth 4 points) A monopolistic seller has a single indivisible good for sale. There are two potential buyers (we will also call them bidders). Each buyer $i = 1, 2$ has value $v_i$ for the auctioned good. Each $v_i$ is an independent draw from the uniform distribution on the interval $[4, 10]$. The good is sold in the second-price sealed bid auction with reserve prices $R \in [4, 10]$.

1. Given reserve price $R$, what is the expected revenue of the seller? Please calculate the expected revenue for $R = 3$, up to two decimal points.

Denote the expected payoff of the seller when the reserve price is $R$ by $E[\Pi(R)]$. As in class, the expected revenue of the seller as follows:

$$E[\Pi(R)] = P(b_1, b_2 > R) \times E[\Pi(R)|b_1, b_2 > R] + P(b_1 > R > b_2 \text{ or } b_2 > R > b_1) \times R$$

By independence of values, and the fact that in a second price auction, bidders bid their true values,

$$P(b_1, b_2 > R) = P(v_i > R)^2 = \left(\frac{10 - R}{6}\right)^2$$

When both bids are above $R$, the expected payment is the expected value of the second highest bid, given that both are independently, uniformly distributed in $[R, 10]$. Same argument as in Part 3, question 2, gives the expected payment to be $\frac{2+1-2}{3} (10 - R) + R = \frac{2}{3} (5 + R)$.

Now,

$$P(b_1 > R > b_2 \text{ or } b_2 > R > b_1) = 2 \times P(v_i < R) \times P(v_j > R) = 2 \times \frac{R - 4}{6} \times \frac{10 - R}{6}$$

Putting these terms together, we obtain the following expression:

$$E[\Pi(R)] = \left(\frac{10 - R}{6}\right)^2 \times \left(\frac{2(5 + R)}{3}\right) + 2 \times \frac{R - 4}{6} \times \frac{10 - R}{6} \times R$$

$$= (10 - R) \left(\frac{1}{6}\right)^2 \left[ (10 - R) \times 2\left(\frac{5 + R}{3}\right) + 2(R - 4)R \right]$$

$$= (10 - R) \left(\frac{1}{6}\right)^2 \left[ \frac{4R^2}{3} - \frac{14R}{3} + \frac{10^2}{3} \right] = (10 - R) \left(\frac{1}{6}\right)^2 \left(\frac{1}{3}\right) \left[ 4R^2 - 14R + 10^2 \right]$$

Now when we set the reserve price of $R = 3$ this reserve price is not binding, and hence the revenue is the same as it would be with any other nonbinding reserve price including $R = 4$. This gives us

$$E[\Pi(3)] = E[\Pi(4)] = (10 - 4) \left(\frac{1}{6}\right)^2 \left(\frac{1}{3}\right) \left[ 4 \times 4^2 - 14 \times 4 + 10^2 \right] = \left(\frac{1}{6}\right) \left(\frac{1}{3}\right) \times [4 \times 4^2 - 14 \times 4 + 100] = 6.$$
2. **Find the revenue-maximizing reserve price \( R \), and provide the argument.**

The revenue maximizing reserve price, call it \( R^* \), is the value of \( R \) that maximizes the above expression for \( E[\Pi(R)] \). The maximum is either at the boundary \( R = 4 \) or \( R = 10 \), or it solves the first order condition

\[
\frac{dE[\Pi(R)]}{dR} = 12 \times \left( \frac{1}{6} \right)^2 \left( \frac{1}{3} \right) \left[ 9R - R^2 - 20 \right] = 0
\]

Notice that we can factorize the quadratic equation in \( R \) into the following:

\[-(R - 5)(R - 4) = 0,\]

hence, there are two solutions for \( R^* \): \( R^*_1 = 5 \) and \( R^*_2 = 4 \), the second one of which is equal to one of the boundaries. To see which is a maximum, we evaluate seller’s expected payoff at each and see which yields a higher payoff. We calculated above the expected revenue for \( R = 4 \) getting \( E[\Pi(4)] = 6 \). For \( R = 10 \) the expected revenue is zero. Finally, for \( R = 5 \), the expected revenue is \( E[\Pi(5)] = 6.019 \). We conclude that \( R^* = 5 \).

3. **Now, suppose that the seller learns that the value of bidder 1 is 5 and the value of bidder 2 is 8. Given this information, what is the revenue-maximizing reserve price? What is the seller’s expected revenue at the revenue maximizing reserve price?**

If the seller knows the bidders’ values, they set the reserve price to be the highest of the two bidders’ values. In this case, they set \( R = 8 \). Their expected revenue is 8, since this is the bid of bidder 2.