Midterm exam PS 30 November 2015

This is a closed book exam. The only thing you can take into this exam is yourself and writing instruments. No calculators, computers, cell phones, etc. are allowed. Everything you write should be your own work. Cases of academic dishonesty will be referred to the Dean of Students office, which has the power to suspend and expel students. Partial credit will be given: math mistakes will not jeopardize your grade. Please show all steps of your work and explain what you are doing at each step. Correct answers alone are worth nothing without a clear and correct explanation of where the answers come from. Clarity and legibility are factors in the grade.

This exam has four parts. Each part is weighted equally (12 points each). Each part is stapled separately (for ease in grading, since the parts are graded separately). Please make sure that your name and student ID number is on each part.

If you have a question, raise your hand and hold up the number of fingers which corresponds to the part you have questions about (if you have a question on Part 2, hold up two fingers). If you need to leave the room during the exam (to use the restroom for example), you need to sign your name on the restroom log before leaving. You can only leave the room once.

When the end of the exam is announced, please stop working immediately. The exams of people who continue working after the end of the exam is announced will have their scores penalized by 30 percent. Please turn in your exam to your TA. When you hand in your exam, please write your name down on the log. Please write all answers on this exam—if you write on the reverse side of pages, please indicate this clearly. Good luck!

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Part 1. Two people play the game of Battleship on a board with six squares, as shown below.

Player 1 fires a missile into one of the six squares. She has six possible strategies. The following are two examples.

Player 2 places a battleship on the board. The battleship occupies two adjacent squares and can be oriented vertically or horizontally. He has six possible strategies. The following are two examples.

If player 1’s missile lands where player 2’s battleship is, then player 1 wins: player 1 gets payoff 1 and player 2 gets payoff 0. If player 1’s missile misses player 2’s battleship, then player 2 wins: player 1 gets payoff 0 and player 2 gets payoff 1.

For example, here is a situation in which player 1 wins.

Here is a situation in which player 2 wins.

(turn to next page)
a. Represent this game as a strategic form game. (4 points)

\[
\begin{array}{cccccccc}
2a & 2b & 2c & 2d & 2e & 2f \\
\hline
1a & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
1b & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
1c & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
1d & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
1e & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
1f & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} & \text{X} \\
\end{array}
\]

b. Use the method of iterative elimination of strongly or weakly dominated strategies. You should be able to reduce it down to a $2 \times 2$ game (a game in which player 1 has two strategies and player 2 has two strategies) by trying out different orders of elimination. Please write down the order of elimination. (4 points)

- 1d w. dom 1f
- 1b w. dom 1c
- 2y w. dom 2a
- 2b w. dom 2f
- 2e w. dom 2d
- 2e w. dom 2c

Other orders are also possible.
c. Write down the remaining 2 × 2 game and find all (pure strategy and mixed strategy) Nash equilibria of this 2 × 2 game. (4 points)

\[
\begin{array}{c|c|c}
\text{c} & \text{b} & \text{c} \\
\hline
\text{b} & (2,0) & (2,2) \\
\hline
\text{c} & (0,1) & (0,0) \\
\end{array}
\]

\[
\text{\text{No pure strategy NE}}
\]

\[
E_{u_1}(1b) = 2 + 0(1-\rho) = 2 \\
E_{u_1}(1c) = 0 + 1(1-\rho) = 1-\rho \\
E_{u_2}(2b) = 0 \cdot \rho + (1-\rho) = 1-\rho \\
E_{u_2}(2c) = 1 \cdot \rho + 0(1-\rho) = \rho
\]

Mixed NE:

\[
\left( 1 \text{ plays } \begin{pmatrix} \frac{1}{2} \end{pmatrix} \text{ with prob } \frac{1}{2}, \quad 2 \text{ plays } \begin{pmatrix} \frac{1}{2} \end{pmatrix} \text{ with prob } \frac{1}{2} \right)
\]
Part 2. Sony Pictures is trying to get Bradley Cooper and Jennifer Lawrence to act in their movie “American Hustle.” Sony will pay Cooper and Lawrence out of its expected profit of $25 million. Sony can either make a fair offer to Cooper and Lawrence, paying them $8 million each (which would leave $9 million for Sony) or make an unfair offer of $9 million to Cooper and $5 million to Lawrence (which leaves $11 million for Sony).

If Sony makes a fair offer, then the game ends (we assume that Cooper and Lawrence both accept). If Sony makes an unfair offer, then Cooper can either reveal the unfair offer to the press or not.

If Cooper reveals it, then the game is over and Sony is very embarrassed and gets a payoff of -$10 million (because it can no longer recruit top women actors), Cooper feels righteous, which is worth $1 million to him, and Lawrence feels good about Cooper’s gesture, which is worth $1 million to her. If Cooper does not reveal that the offer is unfair, then Lawrence chooses whether to accept it or reject it. If Lawrence accepts it, then everyone gets paid according to the unfair offer. If Lawrence rejects it, then no movie is made and everyone gets 0.

a. Model this as an extensive form game. Please write down your payoffs as (Sony, Cooper, Lawrence). (3 points)

b. Find all subgame perfect Nash equilibria of this game. (3 points)
c. Convert your extensive form game into a strategic form game. Find all pure strategy Nash equilibria. (3 points)

\[
\begin{array}{c|cc}
& \text{Cooper} & \\
\text{Sony} & \text{reveal} & \text{not} \\
\hline
\text{fair} & 9,8,8 & 9,8,8 \\
\text{unfair} & -10,1,1 & 11,9,5 \\
\end{array}
\]

\[
\begin{array}{c|cc}
& \text{Cooper} & \\
\text{J-Law} & \text{reveal} & \text{not} \\
\hline
\text{fair} & 9,8,8 & 9,8,8 \\
\text{unfair} & -10,1,1 & 0,0,0 \\
\end{array}
\]

pure strat ME:

\( (\text{fair, reveal, accept}) \) \quad \( (\text{fair, reveal, not}) \) \\
\( (\text{unfair, not, accept}) \) \quad \( (\text{fair, not, not}) \)

d. Now say that Lawrence gets an offer of \( x \) million dollars from Columbia Pictures to star in “X-Men Apocalypse.” The game is exactly the same as in part a. above, except that Lawrence’s payoff from rejecting the unfair offer is now different. If Lawrence rejects the unfair offer, now she gets \( x \) million (Cooper and Sony still get 0 because “American Hustle” does not get made). Circle the values of \( x \) below that guarantee that Sony will make a fair offer (in other words, in all Nash equilibria of the game, Sony makes a fair offer). (3 points)

\[
\begin{array}{cccccccc}
& x = 1 & x = 2 & x = 3 & x = 4 & x = 5 & x = 6 & x = 7 & x = 8 & x = 9 \\
\hline
\text{Sony} & 9,1,8 & 9,1,1 & 11,9,5 & & & & & & \\
\text{Cooper} & -10,1,1 & & & & & & & & \\
\text{J-Law} & -10,1,1 & & & & & & & & \\
\end{array}
\]

If \( x \geq 6 \), J-Law will reject. Hence Cooper will reveal and thus Sony will make a fair offer.

If \( x = 5 \), there is still a SPNE in which J-Law accepts and thus Cooper does not reveal, and thus Sony makes an unfair offer. So \( x = 5 \) does not “guarantee” that Sony makes a fair offer.
Part 3. In 1993, North Korea (NK) and the United States (US) came close to a nuclear crisis. North Korea announced its intention to produce a nuclear weapon and withdraw from the Nonproliferation Treaty, and the Clinton administration began serious preparations for a preemptive attack.

Say that North Korea decides whether to shut down its nuclear program or continue. The US decides whether or not to carry out an attack. They choose simultaneously.

For North Korea, the best outcome is to continue while the US does not attack. The next best outcome is to continue while the US attacks; although its main facilities will be destroyed, NK can still hope to become a nuclear power. The third best outcome is to shut down while the US does not attack, because once its nuclear program is shut down, it is very costly to restart again. Finally, the worst outcome is to shut down while the US attacks. For NK’s payoffs, use 200 for the best, 50 for the second best, -50 for the third best and -100 for the worst.

The best outcome for the US is to not attack while NK shuts down, which achieves nonproliferation at no cost. The next best outcome for the US is to attack while NK shuts down; nonproliferation is achieved, but the US bears the censure of the international community. The next preferable outcome for the US is to attack while NK continues its nuclear project. Finally, the worst outcome is to not attack while NK continues. For the US’s payoffs, use 100 for the best, 10 for the second best, -10 for the third best and -100 for the worst.

a. Model this as a strategic form game and find all Nash Equilibria (pure strategy and mixed strategy). Make North Korea player 1. (2 points)

\[
\begin{array}{cc|cc}
& \text{US} & \text{attack} & \text{not} \\
\hline
\text{NK} & \text{shut down} & -100, 10 & -50, 100 \\
& \text{continue} & 50, -10 & 200, -100 \\
\end{array}
\]

pure strat NE: (continue, attack)

For NK, continue s. dominates shut down

given this, for US, attacking s. dominates not attacking.

So there is no mixed strat NE.
b. Now former President Jimmy Carter travels to NK to dissuade its leader Kim Il-Sung in person from pursuing the nuclear project. The presence of a distinguished foreign visitor provides an opportunity for NK to compromise and back down without losing face. North Korea’s payoffs for shutting down increase by \( x \), where \( x \) is a positive number. Write down this new game. (2 points)

\[
\begin{array}{c|cc}
\text{US} & \text{attack} & \text{not} \\
\hline
\text{shut down} & -100 + x, 10 & -50 + x, 100 \\
\text{continue} & 50, -10 & 200, -100 \\
\end{array}
\]

c. Depending on the value of \( x \), is it possible for this game to have exactly two pure strategy Nash Equilibria? Why or why not? (2 points)

To make the US’s best response, if there are two pure NE, they would have to be \((\text{continue, attack})\) and \((\text{shut down, not})\). If \((\text{continue, attack})\) is a NE, we must have \( 50 > -100 + x \), or \(-50 + x \geq 200\).

Hence we must have \( 50 > x \) and \( x > 250 \), which is impossible.

d. The happiest outcome for the international community is if NK shuts down and the US does not attack. Circle the values of \( x \) below which make this happiest outcome the only (pure strategy) Nash equilibrium. (2 points)

\[
\begin{array}{ccccccc}
x = 98 & x = 99 & x = 100 & x = 101 & x = 102 \\
x = 148 & x = 149 & x = 150 & x = 151 & x = 152 \\
x = 198 & x = 199 & x = 200 & x = 201 & x = 202 \\
x = 248 & x = 249 & x = 250 & x = 251 & x = 252 \\
x = 298 & x = 299 & x = 300 & x = 301 & x = 302
\end{array}
\]

To make \((\text{shut down, not})\) a NE, we must have \(-50 + x \geq 200\), in other words \( x \geq 250 \). Since there is never more than one pure strategy NE in this game, when \( x \geq 250 \), the only pure strategy NE is \((\text{shut down, not})\).
e. Now say that (in addition to the Carter effect above) China promises the US that it will dissuade NK from pursuing its nuclear ambitions. Chinese involvement increases the US payoffs from not attacking by $y$, where $y$ is a positive number.

Now say that NK and the US move sequentially. First, NK decides whether to shut down or continue. Then the US decides whether to attack or not. (In other words, if NK shuts down, the US decides whether to attack or not; if NK continues, the US decides whether to attack or not.) Given the Carter and China factors, model this as an extensive form game. (2 points)

\[
\begin{array}{ccc}
\text{Shut Down} & \text{Continue} \\
\text{US} & & \\
\text{Attack} & (-100+x, 10) & (-50+x, 100+y) \\
\text{Not} & (50, -10) & (200, -100+y)
\end{array}
\]

f. The subgame perfect Nash equilibria of this game depend on the values of $x$ and $y$. The unhappiest outcome for the international community is if NK continues its nuclear program and the US attacks. Circle the values of $x$ and $y$ below which make it possible that this unhappiest outcome occurs in a subgame perfect Nash equilibrium. (2 points)

\[
\begin{array}{cccccccc}
\text{x} &=& 98 & 99 & 100 & 101 & 102 \\
\text{y} &=& 48 & 49 & 50 & 51 & 52 \\
\text{y} &=& 88 & 89 & 90 & 91 & 92 \\
\text{y} &=& 98 & 99 & 100 & 101 & 102
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{x} &=& 148 & 149 & 150 & 151 & 152 \\
\text{y} &=& 48 & 49 & 50 & 51 & 52 \\
\text{y} &=& 88 & 89 & 90 & 91 & 92 \\
\text{y} &=& 98 & 99 & 100 & 101 & 102
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{x} &=& 198 & 199 & 200 & 201 & 202 \\
\text{y} &=& 48 & 49 & 50 & 51 & 52 \\
\text{y} &=& 88 & 89 & 90 & 91 & 92 \\
\text{y} &=& 98 & 99 & 100 & 101 & 102
\end{array}
\]

For NK to continue here, must have \[50 \geq 50 + x\], in other words \[100 \geq x\].

When \[x = 100\] or \[y = 90\], there exist other SPNE in which NK does not continue or US does not attack, but it is still possible (there exists a SPNE) that NK continues and US attacks.
Part 4. Say that Alberto and Bonnie are playing a game which goes like this. Each person chooses a number simultaneously. Alberto's payoffs are as follows. If the sum of the two numbers is a multiple of 2 (in other words, an even number), then Alberto's payoff is the sum of the two numbers. If the sum of the two numbers is not a multiple of 2 (in other words, an odd number), the Alberto gets payoff 0. Bonnie's payoffs are as follows. If the sum of the two numbers is a multiple of 3, then Bonnie's payoff is the sum of the two numbers. If the sum of the two numbers is not a multiple of 3, then Bonnie gets 0.

For example, if Alberto chooses 4 and Bonnie chooses 5, then the sum of the two numbers is 9 and Alberto gets payoff 0 and Bonnie gets payoff 9. If Alberto chooses 2 and Bonnie chooses 6, then Alberto gets 8 and Bonnie gets 0. If Alberto chooses 3 and Bonnie chooses 3, then they both get 6.

a. Say that Alberto can choose from the numbers 1, 4, and 5 and Bonnie can choose from the numbers 3, 4, and 6. Write this as a strategic form game. (2 points)

b. Make a prediction using iterative elimination of weakly or strongly dominated strategies. Please show the order of domination. (1 point)

1. For B, 4 w. dom 3 and 6.
2. For A, 4 w. dom 1 and 5. Prediction: A plays 4 and B plays 4.

c. Find all pure strategy Nash equilibria of this game. (1 point)
d. Now say that Alberto can choose from the numbers 1, 4, and 5, but now Bonnie can choose from the numbers 5, 6, and 7. Write this as a strategic form game. (2 points)

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<td>6, 0</td>
<td>0, 0</td>
<td>8, 0</td>
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<tr>
<td>4, 1</td>
<td>0, 9</td>
<td>10, 0</td>
<td>0, 10</td>
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<tr>
<td>5, 12</td>
<td>0, 10</td>
<td>12, 12</td>
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e. Make a prediction using iterative elimination of weakly or strongly dominated strategies. Please show the order of domination. (1 point)

1. For A, 5 w. dominates 1.
2. For B, 7 w. dominates 6.
3. For A, 5 w. dominates 4.
4. For B, 7 w. dominates 5.

Prediction: A plays 5
B plays 7

f. Find all pure strategy Nash equilibria of this game. (1 point)

pure strat NE: (5, 7)
g. Now say that Alberto can choose from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10, and Bonnie can choose from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Find all pure strategy Nash equilibria of this game. Please explain your reasoning. (Hint: don’t write out the entire game! Try to solve the problem just by thinking about it.) (4 points)

Regardless of what Bonnie does, Alberto can always make the sum an even number.

Regardless of what Alberto does, Bonnie can always make the sum a multiple of 3.

So in any NE, the sum will be a multiple of 6.

If the sum is 6, we have

\[(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\]

None of these are NE because Alberto will deviate to a number which is two larger.

If the sum is 12, we have

\[(2, 10), (3, 9), (4, 8), (5, 7), (6, 6)\]

A gains by deviating to a number which is 2 larger

B gains by deviating to a number 3 larger

So none of these are NE.

If the sum is 18, we have

\[(8, 10), (9, 9), (10, 8)\]

\[(8, 10)\] is not a NE because A will deviate to 10.

So the only pure strategy NE are \[(9, 9)\] and \[(10, 8)\]