Answers to
Final exam PS 172 March 2014

Name:

This is a closed book exam. The only thing you can take into this exam is yourself and writing instruments. No calculators, computers, cell phones, etc. are allowed. Everything you write should be your own work. Cases of academic dishonesty will be referred to the Dean of Students office, which has the power to suspend and expel students. Partial credit will be given: math mistakes will not jeopardize your grade. This exam has five parts. Each part is weighted equally (12 points each). Please show all steps of your work and explain what you are doing at each step. Correct answers alone are worth nothing without a clear and correct explanation of where the answers come from. Clarity and legibility are factors in the grade.

If you need to leave the room during the exam (to use the restroom for example), you need to sign your name on the restroom log before leaving. You can only leave the room once.

When the end of the exam is announced, please stop working immediately. The exams of people who continue working after the end of the exam is announced will have their scores penalized by 30 percent. When you hand in your exam, please write your name down on the log. Please write all answers on this exam—if you write on the reverse side of pages, please indicate this clearly. Good luck!
1. Say that person 1 and person 2 each choose whether to arm \((a)\) or not \((n)\). But they are not sure what game they are playing: it could be either Chicken, Coordination, or Prisoners’ Dilemma. Each of these three possibilities occurs with probability \(1/3\), and the payoffs are given below.

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(-12, -12)</td>
<td>(6, -3)</td>
</tr>
<tr>
<td>(n)</td>
<td>(-3, 6)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

\[\text{Chicken}\]

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(6, 6)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>(n)</td>
<td>(0, 0)</td>
<td>(12, 12)</td>
</tr>
</tbody>
</table>

\[\text{Coordination}\]

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(0, 0)</td>
<td>(12, -6)</td>
</tr>
<tr>
<td>(n)</td>
<td>(-6, 12)</td>
<td>(3, 3)</td>
</tr>
</tbody>
</table>

\[\text{Prisoners’Dilemma}\]

a. Say that person 1 and person 2 are completely ignorant of what game they are playing. Find all Bayesian Nash equilibria of this game. (2 points)

\[\text{NE: } (a a, a n, n n)\]

b. Now say that person 1 and person 2 both always know exactly what game they are playing. Find all Bayesian Nash equilibria. (It is of course OK to solve this problem by setting up a very large game, but you can make a simplification which makes it somewhat easier—if you make the simplification, please explain clearly why you can.) (2 points)

\[\text{In the Prisoners’ Dilemma,}\]

\[a\] s. dominates \(n\). So if you know it’s a Prisoners’ Dilemma, you will play \(a\).

So we only have to consider 4 strategies for each player.

\[\text{NE: } (n n, a n, n a, a a)\]

\[\text{NE: } (a a, a n, n n, n a)\]

\[\text{NE: } (a a, n a, a a, n n)\]

\[\text{NE: } (n a, a n, a a, n n)\]
Here is the game again for your reference.

<table>
<thead>
<tr>
<th></th>
<th>Chicken</th>
<th>Coordination</th>
<th>Prisoners’ Dilemma</th>
</tr>
</thead>
<tbody>
<tr>
<td>a  n</td>
<td>-12, -12</td>
<td>6, -3</td>
<td>a  n</td>
</tr>
<tr>
<td>a  n</td>
<td>6, 6</td>
<td>0, 0</td>
<td>a  n</td>
</tr>
<tr>
<td>a  n</td>
<td>-6, 12</td>
<td>3, 3</td>
<td></td>
</tr>
</tbody>
</table>

C. Now say that person 1 knows whether they are playing Chicken or not, but does not know whether they are playing Coordination or Prisoners’ Dilemma. Person 2 is completely ignorant. Find all Bayesian Nash equilibria. (2 points)

D. Again, person 1 knows whether they are playing Chicken or not, but does not know whether they are playing Coordination or Prisoners’ Dilemma. Now person 2 knows whether they are playing Prisoners’ Dilemma or not, but does not know whether they are playing Chicken or Coordination. Find all Bayesian Nash equilibria. (2 points)
e. Among the four scenarios above (a., b., c., and d.), does person 2 like one scenario best of all? If so, write down the scenario that person 2 likes best and explain why. If not, explain why not. (2 points)

**Person 2 likes scenario d. best -**
She gets at least 4 and maybe 7.

f. Here is the game again for your reference.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>n</th>
<th></th>
<th>a</th>
<th>n</th>
<th></th>
<th>a</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>−12, −12</td>
<td>6, −3</td>
<td>a</td>
<td>6, 6</td>
<td>0, 0</td>
<td>a</td>
<td>0, 0</td>
<td>12, −6</td>
</tr>
<tr>
<td>n</td>
<td>−3, 6</td>
<td>0, 0</td>
<td>n</td>
<td>0, 0</td>
<td>12, 12</td>
<td>n</td>
<td>−6, 12</td>
<td>3, 3</td>
</tr>
</tbody>
</table>

**Chicken**  
**Coordination**  
**Prisoners’ Dilemma**

Note that the highest possible payoff for person 2 is 6 in Chicken, 12 in Coordination, and 12 in Prisoners’ Dilemma. Since each game occurs with probability 1/3, the highest possible expected payoff for person 2 is \((1/3)6 + (1/3)12 + (1/3)12 = 10\).

Say that you can make the information partitions of person 1 and person 2 anything that you want. Is it possible to write down information partitions to make it so that there exists a Bayesian Nash equilibrium in which person 2 gets 10, her highest possible payoff? If so, write down the information partitions and the Bayesian Nash equilibrium in which person 2 gets 10. If not, explain why not. (Obviously, don’t try to solve this problem through brute force, i.e. writing down a game for all possible knowledge partitions—just try to reason it out.) (2 points)

1 plays \( n \)  
2 plays \( a \)  

\( \begin{array}{ccc}
-12 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
-3 & 12 & -6
\end{array} \) if 1 deviates to a

12 12 12 \( \leq 2 \)’s highest possible payoffs

To “make” 1 not want to deviate to \( a \), we can make his partition \( \circ \circ \circ \) or \( \circ \circ \circ \).  
2 will not deviate because she is getting her highest possible payoff.  
Her partition can be \( \circ \circ \circ \circ \circ \) or \( \circ \circ \circ \).
2. Say that the city council of Bronx Beach is controlled by three political parties. Party A controls 4 seats in the council, party B controls 3 seats, and party C controls 1 seat. There are a total of 8 seats in the council, and passing a bill requires a majority. A majority means strictly greater than half; since there are 8 seats in total, a majority is 5 seats. The city council decides how to split one dollar. Since parties A and B together control 7 seats, they can create a majority, and we can write \( v(\{A, B\}) = 1 \). Similarly, party A and party C together have 5 seats and can create a majority, and we write \( v(\{A, C\}) = 1 \). But parties B and C together have only 4 seats and do not have a majority, and so we write \( v(\{B, C\}) = 0 \).

a. Find the Shapley value of this game. (3 points)

\[
\begin{array}{ccc}
\text{A} & \text{B} & \text{C} \\
\hline
\text{A} & 0 & 1 & 0 \\
\text{B} & 1 & 0 & 0 \\
\text{C} & 1 & 0 & 0 \\
\text{A} & 1 & 1 & 0 \\
\end{array}
\]

\[
\frac{4}{6}, \frac{4}{6}, \frac{1}{6}, \frac{1}{6}
\]

Shapley value

\[
\left( \frac{4}{6}, \frac{4}{6}, \frac{1}{6}, \frac{1}{6} \right)
\]

b. Now say that there are four parties. Party A controls 4 seats, party B controls 3 seats, party C controls 1 seat, and party D controls 1 seat. Since there are a total of 9 seats in the council, a majority requires 5 seats. Find the Shapley value of this game. (3 points)

A contributes when

\[
\begin{array}{ccc}
\text{A} & \text{B} & \text{D} \\
\hline
\text{A} & 6 & 6 \\
\end{array}
\]

So Shapley value is

\[
\left( \frac{12}{24}, \frac{9}{24}, \frac{4}{24}, \frac{4}{24} \right)
\]

\[
= \left( \frac{1}{2}, \frac{1}{8}, \frac{1}{6}, \frac{1}{6} \right)
\]
c. Now say that party A controls 4 seats, party B controls 4 seats, party C controls 1 seat, and party D controls 1 seat. Since there are a total of 10 seats in the council, a majority requires 6 seats. Find the Shapley value of this game. (2 points)

\[
\begin{align*}
4! &= 24 \text{ orders} \quad 4 \quad 4 \quad 1 \quad 1 \\
A \text{ contributes when} \quad C \text{ contributes when} \\
BA &= 2 \\
\text{} & \quad A \quad DC \quad B \\
\text{} & \quad D \quad AC \quad B \\
\text{} & \quad B \quad DC \quad A \\
\text{} & \quad D \quad BC \quad A \\
\text{} & \quad \frac{1}{24} \\
\text{Shapley value} \quad \left( \frac{3}{24}, \frac{8}{24}, \frac{4}{24}, \frac{4}{24} \right) \\
&= \left( \frac{1}{8}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6} \right)
\end{align*}
\]

d. Now say that there are five parties. Party A controls 4 seats, party B controls 3 seats, party C controls 1 seat, party D controls 1 seat, and party E controls 1 seat. Since there are a total of 10 seats in the council, a majority requires 6 seats. Find the Shapley value of this game. (2 points)

\[
\begin{align*}
5! &= 120 \text{ orders} \quad A \quad B \quad C \quad D \quad E \\
\text{} & \quad 4 \quad 3 \quad 1 \quad 1 \quad 1 \\
A \text{ contributes when} \quad B \text{ contributes when} \quad C \text{ contributes when} \\
BA &= 6 \\
\text{} & \quad A \quad B \quad C \\
\text{} & \quad A \quad B \quad C \\
\text{} & \quad B \quad C \quad A \\
\text{} & \quad B \quad C \quad A \\
\text{} & \quad C \quad A \quad E \\
\text{} & \quad C \quad A \quad E \\
\text{} & \quad \frac{24}{54} \\
\text{Shapley value} \quad \left( \frac{54}{120}, \frac{24}{120}, \frac{14}{120}, \frac{14}{120}, \frac{14}{120} \right) \\
&= \left( \frac{9}{20}, \frac{1}{5}, \frac{7}{60}, \frac{7}{60}, \frac{7}{60} \right)
\end{align*}
\]
e. Now say that party A controls 4 seats, party B controls 4 seats, party C controls 1 seat, party D controls 1 seat, and party E controls 1 seat. Since there are a total of 11 seats in the council, a majority requires 6 seats. Find the Shapley value of this game. (2 points)

\[ 5! = 120 \text{ orders} \]

A contributes when

\[ BA \quad 24 \]

\[ \quad \quad A \quad 24 \]

\[ \quad \quad \quad A \quad B \quad 6 \quad \frac{6}{36} \]

C contributes when

\[ A \quad C \quad B \quad 2 \]

\[ A \quad C \quad B \quad 2 \]

\[ \quad \quad A \quad C \quad B \quad 2 \]

\[ \quad \quad \quad A \quad C \quad B \quad 2 \]

\[ \quad \quad \quad \quad A \quad C \quad B \quad 2 \]

\[ \quad \quad \quad \quad \quad A \quad C \quad B \quad 2 \]

\[ \quad \quad \quad \quad \quad \quad B \quad C \quad A \quad 2 \]

\[ \quad \quad \quad \quad \quad \quad \quad B \quad C \quad A \quad 2 \]

\[ \quad \quad \quad \quad \quad \quad \quad \quad B \quad C \quad A \quad 2 \]

Shapley value

\[ \left( \frac{36}{120}, \frac{36}{120}, \frac{16}{120}, \frac{16}{120}, \frac{16}{120} \right) \]

\[ = \left( \frac{3}{10}, \frac{3}{10}, \frac{2}{15}, \frac{2}{15}, \frac{1}{15} \right) \]
3. Say that two people are deciding how many minutes to spend at a cocktail party. Person 1 chooses to spend \(a_1\) minutes at the party, and person 2 chooses to spend \(a_2\) minutes at the party. A person is either gregarious \((g)\) or reclusive \((r)\). If person 1 is gregarious, her utility function is \(u_1(g, a_1, a_2) = (120 + a_2)a_1 - (a_1)^2\). If person 1 is reclusive, her utility function is \(u_1(r, a_1, a_2) = (120 - a_2)a_1 - (a_1)^2\). In other words, if person 1 is gregarious, her utility increases when person 2 spends more time at the party; if person 1 is reclusive, her utility decreases when person 2 spends more time at the party.

Similarly, if person 2 is gregarious, his utility function is \(u_2(g, a_1, a_2) = (120 + a_1)a_2 - (a_2)^2\). If person 2 is reclusive, his utility function is \(u_2(r, a_1, a_2) = (120 - a_1)a_2 - (a_2)^2\).

a. Say that person 1 is always gregarious. Person 2 is either gregarious or reclusive; he is gregarious with probability \(1/2\) and reclusive with probability \(1/2\). (In other words, there is no uncertainty about person 1; the only uncertainty is about person 2.) Person 2 of course knows whether he himself is gregarious or reclusive. Person 1 also knows whether person 2 is gregarious or reclusive. Find the Bayesian Nash equilibrium of this game. (3 points)

\[
\begin{align*}
\text{EU}_1 &= \frac{1}{2} \left( (120 + a_2) a_1^g - (a_1^g)^2 \right) + \frac{1}{2} \left( (120 + a_2^r) a_1^r - (a_1^r)^2 \right) \\
\text{EU}_2 &= \frac{1}{2} \left( (120 + a_1^g) a_2^g - (a_2^g)^2 \right) + \frac{1}{2} \left( (120 - a_1^r) a_2^r - (a_2^r)^2 \right)
\end{align*}
\]

So NE is \((120, 72), (120, 24)\).
b. Again, say that person 1 is always gregarious. Person 2 is either gregarious or reclusive; he is gregarious with probability 1/2 and reclusive with probability 1/2. (In other words, there is no uncertainty about person 1; the only uncertainty is about person 2.) Person 2 of course knows whether he himself is gregarious or reclusive. Now say that person 1 does not know whether person 2 is gregarious or reclusive. Find the Bayesian Nash equilibrium of this game. (3 points)

Here are the utility functions again for your convenience:

\[
\begin{align*}
  u_1(g, a_1, a_2) &= (120 + a_2)a_1 - (a_1)^2 \\
  u_2(g, a_1, a_2) &= (120 + a_1)a_2 - (a_2)^2 \\
  u_1(r, a_1, a_2) &= (120 - a_2)a_1 - (a_1)^2 \\
  u_2(r, a_1, a_2) &= (120 - a_1)a_2 - (a_2)^2
\end{align*}
\]

**For part a, we know**

\[
\begin{align*}
  a_2^g &= 60 + \frac{a_1}{2} \quad \text{and} \quad a_2^r = 60 - \frac{a_1}{2} \\
  \text{So} \quad 240 + 60 + \frac{a_1}{2} + 60 - \frac{a_1}{2} &= 4a_1 \\
  360 &= 4a_1 \\
  a_1 &= 90 \\
  \text{Hence} \quad a_1^g &= 60 + \frac{90}{2} = 105 \quad a_2^r = 60 - \frac{90}{2} = 15
\end{align*}
\]

**The NE is**

\[(g_0, 20), (105, 15)\]
c. Now say that person 1 is either gregarious or reclusive and person 2 is either gregarious or reclusive. In other words, now there is uncertainty about both people. Each person is gregarious with probability 1/2 and reclusive with probability 1/2. Therefore there are now four states of the world, each occurring with probability 1/4. Say that person 1 knows whether she herself is gregarious or reclusive but not whether person 2 is gregarious or reclusive. Similarly, person 2 knows whether he herself is gregarious or reclusive but not whether person 1 is gregarious or reclusive. Find the Bayesian Nash equilibrium of this game. [It is OK to employ symmetry arguments when solving for the equilibrium (i.e. person 1 and person 2 are symmetric) if you find this helps—just explain what you are doing and be careful.] (3 points)

Here are the utility functions again for your convenience:

\[
\begin{align*}
 u_1(g, a_1, a_2) &= (120 + a_2)a_1 - (a_1)^2 \\
 u_2(g, a_1, a_2) &= (120 + a_1)a_2 - (a_2)^2 \\
 u_1(r, a_1, a_2) &= (120 - a_2)a_1 - (a_1)^2 \\
 u_2(r, a_1, a_2) &= (120 - a_1)a_2 - (a_2)^2
\end{align*}
\]

\[
\begin{array}{c|cc}
 & g & r \\
\hline
 s & a_1^g & a_1^r \\
 r & a_2^g & a_2^r
\end{array}
\]

\[
\begin{align*}
 Eu_1 &= \frac{1}{4} \left( (120 + a_2^g) a_1^g - (a_1^g)^2 + \frac{1}{4} \left( (120 + a_2^r) a_1^r - (a_1^r)^2 \right) \right) \\
 &\quad + \frac{1}{4} \left( (120 - a_2^g) a_1^g - (a_1^g)^2 + \frac{1}{4} \left( (120 - a_2^r) a_1^r - (a_1^r)^2 \right) \right)
\end{align*}
\]

\[
\begin{align*}
 \frac{d Eu_1}{da_1^g} &= \frac{1}{4} \left( (120 + a_2^g - 2a_1^g) + \frac{1}{4} (120 + a_2^r - 2a_1^r) \right) = 0 \\
 240 + a_2^g + a_2^r &= 4a_1^g
\end{align*}
\]

\[
\begin{align*}
 \frac{d Eu_1}{da_1^r} &= \frac{1}{4} \left( (120 - a_2^g - 2a_1^r) + \frac{1}{4} (120 - a_2^r - 2a_1^r) \right) = 0 \\
 240 - a_2^g - a_2^r &= 4a_1^r
\end{align*}
\]

By symmetry, \( a_1^g = a_2^g \) and \( a_1^r = a_2^r \) so we have

\[
\begin{align*}
 240 + a_1^g + a_1^r &= 4a_1^g \\
 240 - a_1^g - a_1^r &= 4a_1^r \\
 480 &= 4a_1^g + 4a_1^r \\
 120 &= a_1^g + a_1^r \\
 360 &= 4a_1^g \\
 90 &= a_1^g \\
 30 &= a_1^r
\end{align*}
\]
d. Again, person 1 is either gregarious or reclusive and person 2 is either gregarious or reclusive. Again, there are four states of the world, each occurring with probability $1/4$. Now both people know everything; both people know whether they themselves are gregarious or reclusive and also whether the other person is gregarious or reclusive. Find the Bayesian Nash equilibrium of this game. (3 points)

Here are the utility functions again for your convenience:

\[
\begin{align*}
    u_1(g, a_1, a_2) &= (120 + a_2)a_1 - (a_1)^2 \\
    u_1(r, a_1, a_2) &= (120 - a_2)a_1 - (a_1)^2 \\
    u_2(g, a_1, a_2) &= (120 + a_1)a_2 - (a_2)^2 \\
    u_2(r, a_1, a_2) &= (120 - a_1)a_2 - (a_2)^2
\end{align*}
\]

From part a, we know

\[
\begin{align*}
    a_1^{gg} &= 120 & a_1^{gr} &= 72 \\
    a_2^{gg} &= 120 & a_2^{sr} &= 24
\end{align*}
\]

By symmetry, $a_1^{rg} = 24$ and $a_2^{rg} = 72$

So we have

\[
\begin{pmatrix}
    120 & 72 \\
    24 & 24 \\
\end{pmatrix}
\]

We just have to solve for $a_1^{rr}$ and $a_2^{rr}$.

\[
\begin{align*}
    u_1 &= (120 - a_2^{rr})a_1^{rr} - (a_1^{rr})^2 \\
    \frac{d u_1}{d a_1^{rr}} &= (120 - a_2^{rr}) - 2a_1^{rr} = 0 \\
    120 - a_1^{rr} &= 2a_2^{rr} = 0 \\
    120 - 2a_2^{rr} &= a_1^{rr} \\
    120 - a_2^{rr} &= 2(120 - 2a_2^{rr}) = 0 \\
    -120 + 3a_2^{rr} &= 0 \\
    a_2^{rr} &= 40 \Rightarrow a_1^{rr} = 120 - 80 = 40
\end{align*}
\]

So NE is

\[
\begin{pmatrix}
    120 & 72 \\
    24 & 40 \\
\end{pmatrix}
\]
4. Consider the two-person game below, in which Nature moves first.

\[
\begin{array}{c|c|c|c|c}
 & f & g & h & i \\
\hline
j & 6,4 & 0,0 & 8,8 & 0,0 \\
\hline
k & 0,0 & 0,0 & 0,0 & 8,8 \\
\hline
l & 8,8 & 8,8 & 8,8 & 8,8 \\
\hline
m & 0,0 & 0,0 & 0,0 & 8,8 \\
\hline
n & 8,8 & 8,8 & 8,8 & 8,8 \\
\hline
\end{array}
\]

a. Find all Nash equilibria of this game. (4 points)
b. Find all Perfect Bayesian Nash equilibria of this game. I write it down several times so you don’t have to spend time writing the trees over and over again. (8 points)
5. Say that persons 1 and 2 are playing a Battle of the Sexes game with an outside option, as shown below.

Here person 1 first chooses whether to stay home (the “outside option”) or go out. If person 1 chooses to go out, then person 1 and person 2 play a Battle of the Sexes game (person 1 chooses either to go to a or b, and person 2 chooses either to go to a or b).

a. Find all Nash equilibria of this game. (3 points)

b. Make a prediction in this game by iteratively eliminating weakly and strongly dominated strategies (try to eliminate as much as possible). (2 points)
c. Find all Perfect Bayesian Nash equilibria of this game. I write it down several times so you don’t have to spend time writing the trees over and over again. (3 points)
Now say that instead of choosing whether to stay home or not, person 1 can choose whether to burn a dollar bill or not. If person 1 chooses not to burn the dollar, then persons 1 and 2 play Battle of the Sexes as before. If person 1 burns the dollar bill, it hurts only himself, and then persons 1 and 2 play Battle of the Sexes.

\[
\begin{array}{c|c|c|c|c}
1 & & & & \\
\hline
\text{burn$\;\$1} & \text{not} & & & \\
\hline
\text{a} & \text{b} & \text{a} & \text{b} & \\
\hline
2, 1 & -1, 0 & -1, 0 & 0, 3 & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
1 & & & & \\
\hline
\text{a} & \text{b} & \text{a} & \text{b} & \\
\hline
3, 1 & 0, 0 & 0, 0 & 1, 3 & \\
\end{array}
\]

d. Write this game as a strategic form game. Make a prediction in this game by iteratively eliminating weakly and strongly dominated strategies (try to eliminate as much as possible). (3 points)

e. Do you predict that person 1 will burn the dollar? Does person 1’s ability to burn the dollar enable him to get his preferred outcome in Battle of the Sexes? (1 point)