Game theory basics
PS 30

Definition of a strategic form game
A strategic form game is composed of 3 things:

- A set of \( n \) players, \( N = \{1, \ldots, n\} \).
- For each player \( i \), a set of strategies \( A_i \).
- For each player \( i \), a utility function \( u_i(a_1, a_2, \ldots, a_n) \).

Example

- Say we have 2 people, person 1 and person 2.
- Each person can either give a present or not give a present. Hence \( A_1 = \{\text{give, notgive}\} \) and \( A_2 = \{\text{give, notgive}\} \).
- Assume that giving a present costs 1 but getting a present gives you a reward of 4. So person 1’s utility function is \( u_1(\text{give, give}) = 3 \), \( u_1(\text{give, notgive}) = -1 \), \( u_1(\text{notgive, give}) = 4 \), and \( u_1(\text{notgive, notgive}) = 0 \).
- We can represent this game with the table below. Here person 1 chooses the row and person 2 chooses the column. In each entry of the table are two numbers: the first number is person 1’s payoff and the second number is person 2’s payoff.

<table>
<thead>
<tr>
<th></th>
<th>2 gives</th>
<th>2 doesn’t give</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 gives</td>
<td>3, 3</td>
<td>-1, 4</td>
</tr>
<tr>
<td>1 doesn’t give</td>
<td>4, -1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Prisoners’ Dilemma

- This game is called the Prisoners’ Dilemma. Each person would rather not give because regardless of what the other person does, not giving is better than giving. Hence neither person gives a present. However, both would be better off if they both gave presents.
- Any game can be specified by a table like this one.

How to make a prediction in a strategic form game
There are basically two ways to make a prediction in a strategic form game:

- Eliminating dominated strategies (and “iteratively” eliminating them).
- Finding Nash equilibria (“pure” and “mixed”).

Dominated strategies

- We say that one strategy \( a_i \) “strongly dominates” another \( a_i’ \) if person \( i \)’s utility from playing \( a_i \) is greater than her utility from playing \( a_i’ \) regardless of what everyone else does.
- For example, in the Prisoners’ Dilemma, for person 1, not giving strongly dominates giving. If person 2 gives, then not giving has a higher utility for person 1 than giving (4 is greater than 3). If person 2 doesn’t give, then not giving has a higher utility for
person 1 than giving (0 is greater than −1). So regardless of what person 2 does, person 1 would be better off not giving than giving.

- The idea is that if a strategy is strongly dominated, then that person will never play it and hence it can be effectively eliminated from the game.

- We say that one strategy $a_i$ “weakly dominates” another $a'_i$ if person $i$’s utility from playing $a_i$ is always at least as great as her utility from playing $a'_i$ and is sometimes strictly better.

- For example, in the game below, $1a$ strongly dominates $1b$ but $2a$ only weakly dominates $2b$.

```
   2a   2b
1a  7, 3  4, 0
1b  6, 1  0, 1
```

**Iterative elimination of dominated strategies**

- Once one strategy is eliminated because it is weakly or strongly dominated, it is sometimes possible to eliminate more of them. Consider the example below, in which $2b$ strongly dominates $2a$.

```
   2a   2b
1a  7, 3  0, 5
1b  6, 4  2, 9
```

- Since $2b$ strongly dominates $2a$, we can say that person 2 will never play $2a$ and hence we can effectively eliminate $2a$ from the game. We then get the following game.

```
   2b
1a  0, 5
1b  2, 9
```

- In this “reduced” game, $1b$ strongly dominates $1a$ and hence we can predict that person 1 will play $1b$.

- Another way to think about this is the following. Person 2 doesn’t play $2a$ because she is rational. Since person 1 knows that person 2 is rational, person 1 knows that person 2 will play $2b$ and hence person 1 plays $1b$.

**Nash equilibrium (in pure strategies)**

- The main problem with the method of iteratively eliminating dominated strategies is that many games do not have dominated strategies, as for example the “chicken” game below.

```
   2a   2b
1a  3, 3  1, 4
1b  4, 1  0, 0
```
In this situation, the most commonly accepted way to make a prediction is to find Nash equilibria.

Definition: A combination of strategies is a “Nash equilibrium” if no individual person can gain by deviating, assuming that everyone else stays the same.

For example, in the “chicken” game, \((1a, 2a)\) is not a Nash equilibrium because person 1 would rather deviate and play \(1b\) and get 4 instead of 3.

In the “chicken” game, \((1a, 2b)\) is a Nash equilibrium. If person 1 deviates, he would get 0 instead of 1, and if person 2 deviates, she would get 3 instead of 4.

In the “chicken” game, the Nash equilibria are \((1a, 2b)\) and \((1b, 2a)\).

Take another example:

<table>
<thead>
<tr>
<th></th>
<th>2a</th>
<th>2b</th>
<th>2c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>5, 3</td>
<td>1, 0</td>
<td>7, 2</td>
</tr>
<tr>
<td>1b</td>
<td>4, 1</td>
<td>0, 0</td>
<td>3, 1</td>
</tr>
<tr>
<td>1c</td>
<td>5, 6</td>
<td>2, 6</td>
<td>4, 4</td>
</tr>
</tbody>
</table>

Check if \((1a, 2a)\) is a Nash equilibrium: if person 1 deviates, he would get either 4 or 5, which is not better than the 5 he is getting; if person 2 deviates, she would get 0 or 2, which is not better than the 3 she is getting. Hence \((1a, 2a)\) is a Nash equilibrium.

By checking each of the 9 possible strategy combinations, we can find that the Nash equilibria in this game are \((1a, 2a)\), \((1c, 2a)\) and \((1c, 2b)\).

Another way to think about a Nash equilibrium is this: In a Nash equilibrium, each person is playing a best response given what everyone else is doing. For example, in the game above, we can write person 1’s best response using stars (*). If person 2 plays 2a, then person 1’s best response is to play either 1a or 1c. If person 2 plays 2b, then person 1’s best response is to play 1c, and so forth.

<table>
<thead>
<tr>
<th></th>
<th>2a</th>
<th>2b</th>
<th>2c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>5*, 3+</td>
<td>1, 0</td>
<td>7*, 2</td>
</tr>
<tr>
<td>1b</td>
<td>4, 1+</td>
<td>0, 0</td>
<td>3, 1+</td>
</tr>
<tr>
<td>1c</td>
<td>5*, 6+</td>
<td>2*, 6+</td>
<td>4, 4</td>
</tr>
</tbody>
</table>

Similarly, we can write person 2’s best response using plus signs (+). If person 1 plays 1a, then person 2’s best response is to play either 2a. If person 1 plays 1b, then person 2’s best response is to play 2a or 2c, and so forth.

It is easy to see that the Nash equilibria are those strategy combinations which have both stars and daggers.
“Mixed” Nash equilibrium

- Sometimes a game does not have any Nash equilibria, as in the “matching pennies” game. Here from any strategy combination, someone wants to deviate.

\[
\begin{array}{c|cc}
2a & 2b \\
\hline
1a & 1, -1 & -1, 1 \\
1b & -1, 1 & 1, -1 \\
\end{array}
\]

- The traditional thing to do in this case is to consider “mixed strategies,” in other words probabilistic actions. The idea here is that in the matching pennies game for example, instead of person 1 playing 1a all the time or 1b all the time, she will play 1a with some probability and 1b with some probability.

- Say that person 1 plays 1a with probability \(p\) and 1b with probability \(1 - p\). Say that person 2 plays 2a with probability \(q\) and 2b with probability \(1 - q\). We write this below.

\[
\begin{array}{c|cc}
[p] & [1 - q] \\
\hline
2a & 2b \\
[p] & 1a & 1, -1 & -1, 1 \\
[1 - p] & 1b & -1, 1 & 1, -1 \\
\end{array}
\]

- To find the “mixed” Nash equilibria, we write down each person’s best response curve, just as in the Cournot example. Say that person 2 plays 2a with probability \(q\) and plays 2b with probability \(1 - q\). What is person 1’s best response? Given what person 2 does, if person 1 plays 1a, he gets expected utility \((q)(1) + (1 - q)(-1) = 2q - 1\). Given what person 2 does, if person 1 plays 1b, he gets expected utility \((q)(-1) + (1 - q)(1) = 1 - 2q\).

- Thus if \(q\) is close to 1, then person 1’s best response is to play 1a. If \(q\) is close to 0, then person 1’s best response is to play 1b. Depending on what \(q\) is, at some point person 1 “switches” from playing 1a to playing 1b. To find this “switchover probability,” we set \(2q - 1 = 1 - 2q\) and get \(4q = 2\) or \(q = 1/2\). When \(q = 1/2\), person 1 gets the same utility from playing 1a and from playing 1b. So we can graph person 1’s best response curve below.

![Diagram of Player 1's best response curve](image-url)
• What is person 2’s best response? Say that person 1 plays 1a with probability $p$ and plays 1b with probability $1 - p$. Given what person 1 does, if person 2 plays 2a, she gets expected utility $(p)(-1) + (1 - p)(1) = 1 - 2p$. Given what person 1 does, if person 2 plays 2b, she gets expected utility $(p)(1) + (1 - p)(-1) = 2p - 1$.

• Thus if $p$ is close to 1, then person 2’s best response is to play 2b. If $p$ is close to 0, then person 2’s best response is to play 2a. Depending on what $p$ is, at some point person 2 “switches” from playing 2b to playing 2a. To find this “switchover probability,” we set $1 - 2p = 2p - 1$ and get $2 = 4p$ or $p = 1/2$. When $p = 1/2$, person 2 gets the same utility from playing 2a and from playing 2b. So we can graph person 2’s best response curve below.

The mixed Nash equilibrium is where the two best response curves intersect. In this case, they intersect at $p = 1/2$ and $q = 1/2$. Hence in the mixed Nash equilibrium of this game, person 1 plays 1a with probability 1/2 and 1b with probability 1/2, and person 2 plays 2a with probability 1/2 and 2b with probability 1/2.

Extensive form games

• In an extensive form game, people do not move simultaneously (as in strategic form games) but can move in response to each other. For example, say that a kidnapper can either choose to kidnap or not, and if the kidnapper kidnaps, then the family can choose either to pay the ransom or not. We write this as a “game tree” below.
Kidnapper

Family

kidnap

nothing

pay

ransom

not pay

0, 0

-5, -10

5, -5

-5, -10

5, -5

At the end of the tree we write the payoffs for each player. If the kidnapper does nothing, then both get payoffs of 0. If the kidnapper kidnaps and then the family does not pay, then the kidnapper gets a payoff of $-5$ and the family gets a payoff of $-10$. If the kidnapper kidnaps and the family does pay, then the kidnapper gets a payoff of 5 and the family gets a payoff of $-5$.

If we write this as a strategic form game, we get the following.

<table>
<thead>
<tr>
<th></th>
<th>Family</th>
</tr>
</thead>
<tbody>
<tr>
<td>kidnap</td>
<td>5, -5</td>
</tr>
<tr>
<td>nothing</td>
<td>0, 0</td>
</tr>
<tr>
<td>pay ransom</td>
<td>-5, -10</td>
</tr>
<tr>
<td>not pay</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

The Nash equilibria of this game are (kidnap, if kidnap, pay ransom) and (nothing, if kidnap, do not pay).

However, one might say that the Nash equilibrium (nothing, if kidnap, do not pay) is not reasonable because it involves the family making a noncredible threat. In other words, if the game was simply the “subgame” below in which only the family moves, then the family would maximize utility by paying the ransom.

<table>
<thead>
<tr>
<th></th>
<th>Family</th>
</tr>
</thead>
<tbody>
<tr>
<td>pay ransom</td>
<td>5, -5</td>
</tr>
<tr>
<td>not pay</td>
<td>-5, -10</td>
</tr>
</tbody>
</table>

In extensive form games, we say that a Nash equilibrium is “subgame perfect” if play in the game corresponds to a Nash equilibrium in every subgame. In a subgame perfect Nash equilibrium, no player makes a “noncredible threat.”

**Backward induction**

One way to find subgame perfect Nash equilibria is to use “backward induction”: start at the end of the game and work your way backwards. For example, take the donut competition game which is in the homework. There are two donut shops, Crazy Crullers and Grease N’ Sugar. There are five towns which are linked together in a line like this:
A—B—C—D—E. Six people live in each town. The two donut shops decide where to locate: Crazy Crullers goes first and then Grease N’ Sugar. People in a given town will go to the donut shop which is closest to them; if a town is equally distant from the two shops, then half will go to one and half will go to the other.

- It is not too hard to figure out that the game looks like this.

To find the subgame perfect Nash equilibrium by backward induction, look at each of the five subgames at the end and figure out what Grease N’ Sugar will do: its optimal choice is shown in bold. Given Grease N’ Sugar’s actions, then Crazy Crullers’s best choice is to go to town C. So in the subgame perfect Nash equilibrium, Crazy Crullers goes to town C and then Grease N’ Sugar goes to town C.