Midterm exam PS 30 November 2016

This is a closed book exam. The only thing you can take into this exam is yourself and writing instruments. No calculators, computers, cell phones, etc. are allowed. Everything you write should be your own work. Cases of academic dishonesty will be referred to the Dean of Students office, which has the power to suspend and expel students. Partial credit will be given: math mistakes will not jeopardize your grade. Please show all steps of your work and explain what you are doing at each step. Correct answers alone are worth nothing without a clear and correct explanation of where the answers come from. Clarity and legibility are factors in the grade.

This exam has four parts. Each part is weighted equally (12 points each). Each part is stapled separately (for ease in grading, since the parts are graded separately). Please make sure that your name and student ID number is on each part.

If you have a question, raise your hand and hold up the number of fingers which corresponds to the part you have questions about (if you have a question on Part 2, hold up two fingers). If you need to leave the room during the exam (to use the restroom for example), you need to sign your name on the restroom log before leaving. You can only leave the room once.

When the end of the exam is announced, please stop working immediately. The exams of people who continue working after the end of the exam is announced will have their scores penalized by 30 percent. Please turn in your exam to your TA. When you hand in your exam, please write your name down on the log. Please write all answers on this exam—if you write on the reverse side of pages, please indicate this clearly. Good luck!
Part 1. Amita, Ben, and Ceci are friends who come together for a potluck dinner. Each decides whether to bring food (f) or not (n). The dinner will only happen if at least two friends bring food; otherwise, the dinner is cancelled. If the dinner is cancelled, the food is not returned to whoever brought it. Each friend ranks outcomes from best to worst as follows: (i) any outcome where the dinner happens and she does not bring food, (ii) any outcome where the dinner happens and she brings food, (iii) any outcome where the dinner is cancelled and she does not bring food, (iv) any outcome where the dinner is cancelled and she brings food. In other words, the best thing is to have the dinner occur but not bring food yourself, and the worst thing is to bring food yourself only to have the dinner cancelled.

The friends start a group text where they sequentially (first Amita, second Ben, and third Ceci) reveal whether or not they will bring food to the potluck dinner.

a. Ceci works at Google and can always bring free food from work. So assume that Ceci always brings food (and hence is not a player). Hence we have a two player game in which Amita and Ben are the players. Model this as an extensive form game. Use the numbers 0, 2, 4, 6 for payoffs. (2 points)

b. Find a subgame perfect Nash equilibrium (if there is more than one, just write down one of them). (1 point)
c. Write this extensive form game as a strategic form game. (2 points)

\[
\begin{array}{cccc}
A & B \\
\begin{array}{ccc}
& n & f \\
\hline \\
 f & 4,4 & 4,6 \\
n & 6,4 & 2,2 \\
\end{array}
\end{array}
\]

d. Are there any strongly or weakly dominated strategies in this game? If so, make a prediction by iterative elimination of strongly or weakly dominated strategies. Please show the order of elimination. (1 point)

\[
\begin{array}{cccc}
A & B \\
\begin{array}{ccc}
& n & f \\
\hline \\
 f & 4,4 & 4,6 \\
n & 6,4 & 2,2 \\
\end{array}
\end{array}
\]

e. Find all pure strategy Nash equilibria of this strategic form game. (1 point)

\[
\begin{array}{cccc}
A & B \\
\begin{array}{ccc}
& n & f \\
\hline \\
 f & 4,4 & 4,6 \\
n & 6,4 & 2,2 \\
\end{array}
\end{array}
\]
f. Now say that Ceci changes jobs and now works for the mayor’s office, which sadly does not serve food. So we cannot assume that Ceci always brings food. Now we have a three player game in which Amita, Ben, and Ceci are the players. Model this as an extensive form game. Use the numbers 2, 4, 6, 8 for payoffs. (2 points)

g. Find a subgame perfect Nash equilibrium (if there is more than one, just write down one of them). (1 point)
h. Now say that there are 12 friends, and each person decides whether to bring food or not. The dinner happens only if at least 8 friends bring food. Each friend decides simultaneously whether to bring food or not, so we have a strategic form game. Is there a Nash equilibrium in which more than 8 friends bring food? Is there a Nash equilibrium in which exactly 8 friends bring food? Is there a Nash equilibrium in which fewer than 8 friends bring food? Please explain your reasoning. (2 points)

If more than 8 people bring food,
this cannot be a NE because one of the people bringing food can gain by deviating.

If exactly 8 people bring food, then this is a NE because the people bringing food do not want to deviate (because then the dinner would be cancelled), and the people not bringing food do not want to deviate (since they are getting their best possible payoff).

If fewer than 8 people bring food, then there is no dinner and anyone who brings food would rather not.

So the only NE in which fewer than 8 people bring food is when no one brings food. There is no dinner but no individual can deviate by herself and make it happen.
Part 2. Say there are three voters. Each voter chooses between two candidates, A and B. The voters choose a candidate by majority rule, and vote simultaneously. Each voter gets a payoff of 2 if A is elected and 0 otherwise. In addition, candidate B gives a bribe of 1 to everyone who votes for B.

a. Write this as a strategic form game and find all pure strategy Nash equilibria. (4 points)

\[
\begin{array}{cccc}
\text{A} & \text{B} & \\
\text{A} & 2, 2, 2 & 2, 3, 2 \\
\text{B} & 3, 2, 2 & 1, 1, 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{A} & \text{B} & \\
\text{A} & 2, 2, 1 & 0, 1, 1 \\
\text{B} & 1, 1, 0 & 1, 1, 1 \\
\end{array}
\]

\[
\text{NE} = (A, A, A) \quad (B, B, B) \\
(B, A, A) \quad (A, A, B)
\]

b. Now say that voter 3 rebels against voter 1 and does not want to vote the same way as voter 1 does. As before, each voter gets a payoff of 2 if A is elected and 0 otherwise, and candidate B gives a bribe of 1 to everyone who votes for B. But now voter 3 gets an additional “bonus” of 2 if she votes for a different candidate than voter 1 does. Write this as a strategic form game and find all pure strategy Nash equilibria. (4 points)

\[
\begin{array}{cccc}
\text{A} & \text{B} & \\
\text{A} & 2, 2, 2 & 2, 3, 2 \\
\text{B} & 3, 2, 4 & 1, 1, 2 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{A} & \text{B} & \\
\text{A} & 2, 2, 5 & 0, 1, 3 \\
\text{B} & 1, 0, 1 & 1, 1, 1 \\
\end{array}
\]

\[
\text{NE} = (B, A, A) \quad (A, A, B)
\]
c. Now say that voter 3 is hired to be the campaign manager for candidate A and hence will always vote for A. So the only players now are voters 1 and 2. As in part a. above, each voter gets 2 if A is elected and 0 otherwise, and candidate B offers a bribe of 1 to any voter who votes for B. Find all pure strategy and mixed strategy Nash equilibria of this game. (3 points)

\[ \begin{array}{c|c|c}
    & A & B \\
\hline
A & 2, 2 & 2, 3 \\
B & 3, 2 & 1, 1 \\
\end{array} \]

**Pure strategy NE:** (B, A)
(A, B)

\[ E_{u1}(A) = 2 - 2 = 2 \]
\[ E_{u1}(B) = 3 - 1 = 2 \]
\[ E_{u2}(A) = 2 - 2 = 0 \]
\[ E_{u2}(B) = 3 - 1 = 2 \]

\[ 2 = 2 \]
\[ 2 = 2 \]
\[ 0 = 2 \]
\[ 2 = 2 \]

\[ 2 = 2 \]
\[ 2 = 2 \]
\[ 0 = 2 \]
\[ 2 = 2 \]

**Mixed NE:**
\( (1 - \frac{1}{2}, A \text{ with } \frac{1}{2}, 2 - \frac{1}{2}, B \text{ with } \frac{1}{2}, 2 - \frac{1}{2}, A \text{ with } \frac{1}{2}, 0) \)
Part 3. King Stark and Lord Frey consider becoming allies. Each person decides simultaneously whether to propose an alliance or remain independent. Both King Stark and Lord Frey get 10 if they successfully form an alliance; if it works, King Stark will get married with one of Lord Frey’s daughters. If one person proposes an alliance while the other does not, the person proposing the alliance gets \(-5\) (because of the embarrassment of making a failed proposal) and the person who does not gets 5 (because he will have an advantage over the other person when they negotiate in the future). If no one proposes an alliance, both get nothing.

a. Model this as a strategic form game and find all pure strategy and mixed strategy Nash equilibria. (2 points)

\[
\begin{array}{cc|cc}
& \text{Propose} & \text{not} \\
\text{Propose} & 10 & 10 & -5, 5 \\
\text{not} & 5 & -5 & 0, 0
\end{array}
\]

Pure strategy NE: (Propose, Propose)  
(Propose, not) 

\[\begin{align*}
\text{EU}_{\text{Stark}} (\text{Propose}) &= 10 - 5(1 - p) = -5 + 15p \\
\text{EU}_{\text{Stark}} (\text{not}) &= 5 + 0(1 - p) = 5 \\
\text{EU}_{\text{Frey}} (\text{Propose}) &= 10p - 5(1 - p) = -5 + 15p \\
\text{EU}_{\text{Frey}} (\text{not}) &= 5p + 0(1 - p) = 5p
\end{align*}\]

Mixed strategy NE: 
(Stark plays Propose with prob \(p = \frac{1}{3}\), Frey plays Propose with prob \(\frac{2}{3}\))

\[\begin{align*}
-5 + 15p &= 5p \\
10p &= 5 \\
p &= \frac{1}{2}
\end{align*}\]
b. The alliance is formed. However, King Stark falls in love with Talisa, who becomes his Queen, and breaks his marriage pact with Lord Frey. Now King Stark must decide whether to propose a new marriage pact (and thereby keep the alliance) or not. If King Stark makes a new proposal, Lord Frey chooses to either accept or reject this proposal: if Lord Frey accepts the proposal, the alliance survives, and if Lord Frey rejects it, the alliance ends. If King Stark does not make a new proposal, Lord Frey chooses whether to remain independent or forge a new alliance with someone else.

King Stark gets 15 if he makes a new proposal and Lord Frey accepts it. King Stark gets −10 if he makes a new proposal and Lord Frey rejects it. If King Stark does not make a new proposal and Lord Frey becomes independent, King Stark gets 0. However, if King Stark does not make a new proposal and Lord Frey forges a new alliance, King Stark gets −16 because Lord Frey’s new alliance is likely to be against King Stark.

Lord Frey gets 12 if he rejects King Stark’s new proposal because he feels a sense of revenge. Lord Frey gets 7 if he accepts the new proposal because he loses the chance to revenge in the future. If King Stark does not make a new proposal and Lord Frey becomes independent, Lord Frey gets 8. If Lord Frey forges a new alliance, Lord Frey gets 10.

Model this as an extensive form game and find a subgame perfect Nash equilibrium (if there is more than one, just write down one of them). (2 points)

c. Write this extensive form game as a strategic form game and find all pure strategy Nash equilibria of this game. (3 points)
d. Say Tywin Lannister, who is King Stark’s enemy, conspires to break the alliance between King Stark and Lord Frey.

Lannister changes the payoffs in the game in part b. above. Lannister tries to motivate King Stark to make a new proposal to Lord Frey by offering inducements $x$: when King Stark makes a new proposal and Lord Frey rejects it, King Stark’s payoff is $-10 + x$. Lannister also promises secret material benefits $y$ to Lord Frey if he forges a new alliance: if Lord Frey forges a new alliance, Lord Frey gets payoff $10 + y$. All other payoffs are the same as in part b. above. Model this as an extensive form game. (2 points)

![Game Diagram](image)

e. The subgame perfect Nash equilibrium of this game depends on $x$ and $y$. Circle all the combinations of $x$ and $y$ below which ensure that in the resulting subgame perfect Nash equilibrium, King Stark makes a new proposal and Lord Frey rejects it. For example, if when $x = 1$ and $y = 1$, King Stark makes a new proposal and Lord Frey rejects it, circle $x=1$ below. Please explain your work. (3 points)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
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<td>5</td>
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<td>13</td>
<td>-3</td>
<td>13</td>
<td>-3</td>
</tr>
</tbody>
</table>

Please explain your work.
Part 4. Say that Oscar and Penelope are playing a game. Each person chooses a digit, and the resulting digits make a two-digit number. For example, if Oscar chooses 3 and Penelope chooses 4, then the resulting two-digit number is 34. If the resulting two-digit number is a prime number (in other words, a number which is divisible only by itself and 1), then Oscar gets 0, and Penelope gets a payoff equal to her own digit. If the resulting two-digit number is not a prime number, then Oscar gets a payoff equal to his own digit, and Penelope gets 0.

For example, if Oscar chooses 5 and Penelope chooses 3, then the resulting two-digit number is 53 and since 53 is a prime number, Oscar gets 0 and Penelope gets 3. If Oscar chooses 5 and Penelope chooses 1, then the resulting number is 51, which is not a prime number, and Oscar gets 5 and Penelope gets 0.

For your reference, here are the two-digit prime numbers: 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

a. Say that Oscar can choose among the digits 2, 3, 4, 5. Say that Penelope can choose among the digits 2, 3, 4, 5. Write this as a strategic form game and make a prediction using iterative elimination of weakly and strongly dominated strategies. Please show the order of elimination. (4 points)

b. Say that Oscar can choose among the digits 1, 2, 4, 9. Say that Penelope can choose among the digits 1, 3, 7, 9. Write this as a strategic form game and eliminate as much as possible using iterative elimination of weakly and strongly dominated strategies. You should be able to reduce it to a 2 by 2 game (a game in which each person has only two strategies). Please show the order of elimination. (4 points)
c. Find all pure strategy and mixed strategy Nash equilibria of the 2 by 2 game which remains after you do part b. above. (4 points)

\[
\begin{array}{c|c|c}
\vspace{1cm}
 & 2 & 1-2 \\
\hline
2 & 7 & 9 \\
-2 & 0 & 9 \\
\hline
\end{array}
\]

No pure strategy NE

\[
\text{Eu}_{osc}(2) = 2q + 0(1-2) = 2q \\
\text{Eu}_{osc}(1) = 0q + 9(1-2) = 9 - 9q
\]

If \( 2q = 9 - 9q \) \( \Rightarrow \) \( 11q = 9 \) \( \Rightarrow \) \( q = \frac{9}{11} \)

\[
\text{Eu}_{pen}(2) = 0p + 7(1-p) = 7 - 7p \\
\text{Eu}_{pen}(1) = 9p + 0(1-p) = 9p
\]

If \( 7 - 7p = 9p \) \( \Rightarrow \) \( 7 = 16p \) \( \Rightarrow \) \( p = \frac{7}{16} \)

Mixed strategy NE:

\[
(\text{Oscar plays } 2 \text{ with prob } \frac{7}{16}, \text{ Penelope plays } 7 \text{ with prob } \frac{9}{11})
\]