This is a closed book exam. The only thing you can take into this exam is yourself and writing instruments. No calculators, computers, cell phones, etc. are allowed. Everything you write should be your own work. Cases of academic dishonesty will be referred to the Dean of Students office, which has the power to suspend and expel students. Partial credit will be given: math mistakes will not jeopardize your grade. This exam has four parts. Each part is weighted equally (12 points each). Please show all steps of your work and explain what you are doing at each step. Correct answers alone are worth nothing without a clear and correct explanation of where the answers come from. Clarity and legibility are factors in the grade.

If you need to leave the room during the exam (to use the restroom for example), you need to sign your name on the restroom log before leaving. You can only leave the room once.

When the end of the exam is announced, please stop working immediately. The exams of people who continue working after the end of the exam is announced will have their scores penalized by 30 percent. When you hand in your exam, please write your name down on the log. Please write all answers on this exam—if you write on the reverse side of pages, please indicate this clearly. Good luck!
1. Say that person 1 and person 2 each choose whether to play \( a \) or \( b \). But they are not sure what game they are playing: it could be the “left” game shown below or the “right” game. The left game occurs with probability \( p \) and the right game occurs with probability \( 1 - p \), where \( p \in [0, 1] \).

\[
\begin{array}{cc}
  & a & b \\
 a & 6,6 & 0,0 \\
b & 0,0 & 6,6 \\
\end{array}
\]

\[
\begin{array}{cc}
  & a & b \\
 a & 4,4 & 4,0 \\
b & 0,4 & 0,0 \\
\end{array}
\]

\textit{Left} \hspace{1cm} \textit{Right}

a. Say that neither person knows which game they are playing. This is a game with incomplete information. Say that \( p = 1/2 \). Find all Nash equilibria. (3 points)

b. Consider our game with incomplete information (again, neither person knows which game they are playing). Note that if \( p = 0 \), then this game has one Nash equilibrium. If \( p = 1 \), the game has two Nash equilibria. At what value of \( p \) does the game “switch” from having one Nash equilibrium to having two Nash equilibria? Please explain your work. (3 points)
Here is the game again. Again, the left game occurs with probability $p$ and the right game occurs with probability $1 - p$, where $p \in [0, 1]$.

\[
\begin{array}{cc}
\text{Left} & \text{Right} \\
 a & a \\
 b & b \\
\end{array}
\]

\[
\begin{array}{cc}
a & 6,6 & 0,0 & 4,4 & 4,0 \\
b & 0,0 & 6,6 & 0,4 & 0,0 \\
\end{array}
\]

c. Now say that person 1 knows which game they are playing but person 2 does not. This is a game with incomplete information. Say that $p = 1/2$. Find all Nash equilibria. (2 points)

d. Consider our game with incomplete information (again, person 1 knows which game they are playing but person 2 does not). Is it possible for this incomplete information game to have three Nash equilibria? If so, find the value(s) of $p$ which make the game have three Nash equilibria. If not, explain why not. (2 points)

e. Is it possible for this incomplete information game to have two Nash equilibria? If so, find the value(s) of $p$ which make the game have two Nash equilibria. If not, explain why not. (2 points)
2. Consider the two-country game below. Country 1 can either attack by land, attack by sea, or not attack. Country 2 can either defend or not. Country 2 cannot tell, however, whether person 1 attacks by land or by sea.

a. Find all Nash equilibria of this game. (1 point)
b. Find all Perfect Bayesian Nash equilibria of this game. I write it down several times so you don’t have to spend time writing the trees over and over again. (2 points)
c. Now say that country 2 is trying to develop hydraulic plasma technology which works excellently against a sea attack. Nothing would please country 2 more than defending a sea attack using this technology. However, country 1 does not know whether country 2 has developed this technology or not. With probability 1/2, country 2 has not developed this technology, and with probability 1/2, country 2 has successfully developed the technology. So we have the game below.

Note that Nature moves first. The left part of the game is the same as in part a. above, and the right part of the game is when country 2 develops the hydraulic plasma technology.

Find all Nash equilibria of this game. (2 points)
d. Find all Perfect Bayesian Nash equilibria of this game. I write it down several times so you don’t have to spend time writing the trees over and over again. (2 points)
e. Now say that country 1 has spies and knows whether country 2 has developed hydraulic plasma technology. However, country 2 has not received communication from its own laboratory and does not know itself whether it has developed the technology.

Find all Nash equilibria of this game. (2 points)
f. Find all Perfect Bayesian Nash equilibria of this game. I write it down several times so you don’t have to spend time writing the trees over and over again. (2 points)

g. Which of the above three situations (a., c., or e.) is best for country 2? (1 point)
3. Say that making a movie requires the participation of three people: a producer, a screenwriter, and a director. If they make a movie, the movie earns a large profit.

a. Say that Columbia Pictures has three employees. Person 1 is a producer, person 2 is a screenwriter, and person 3 is a director. Use the Shapley value to find each person’s share of the profits. (3 points)

b. Say that Paramount Pictures has four employees. Person 1 is a producer, person 2 is a screenwriter, person 3 is a director, and person 4 is also a director. Use the Shapley value to find each person’s share of the profits. (3 points)
c. Say that TriStar Pictures has five employees. Person 1 is a producer, person 2 is a screenwriter, and persons 3, 4, and 5 are all directors. Use the Shapley value to find each person's share of the profits. (3 points)

d. Say that MGM Studios has five employees. Person 1 is a producer, persons 2 and 3 are screenwriters, and persons 4 and 5 are directors. Use the Shapley value to find each person's share of the profits. (3 points)
4. Say that there are three friends who are each deciding where to locate at the beach. Person 1 chooses location \( a_1 \), person 2 chooses location \( a_2 \), and person 3 chooses location \( a_3 \), where \( a_1, a_2, a_3 \in [0, 1] \). The pier is at position 0 and the lifeguard station is at position 1. Person 1 likes to be close to her friends but also the pier. So person 1’s utility function is

\[
u_1(a_1, a_2, a_3) = -(a_1 - 0)^2 - (a_1 - a_2)^2 - (a_1 - a_3)^2.
\]

Person 2 likes to be close to other people but also the lifeguard station. Thus her utility function is

\[
u_2(a_1, a_2, a_3) = -(a_2 - 1)^2 - (a_2 - a_1)^2 - (a_2 - a_3)^2.
\]

Person 3 likes to be close to other people but also the lifeguard station. Thus his utility function is

\[
u_3(a_1, a_2, a_3) = -(a_3 - 1)^2 - (a_3 - a_1)^2 - (a_3 - a_2)^2.
\]

a. Say that person 3 sprains his ankle and is stuck at \( a_3 = 1 \). Because person 3 cannot move, the situation is a two-person game, in which person 1 and person 2 are the only players. Find the Nash equilibrium of this game. (3 points)
Here are the utility functions again for your convenience.

\[ u_1(a_1, a_2, a_3) = -(a_1 - 0)^2 - (a_1 - a_2)^2 - (a_1 - a_3)^2 \]
\[ u_2(a_1, a_2, a_3) = -(a_2 - 1)^2 - (a_2 - a_1)^2 - (a_2 - a_3)^2 \]
\[ u_3(a_1, a_2, a_3) = -(a_3 - 1)^2 - (a_3 - a_1)^2 - (a_3 - a_2)^2 \]

b. Now say that person 3’s ankle recovers and now person 3 can move around. Now the situation is a three-person game. Find the Nash equilibrium of this game. (3 points)
c. Now say that person 1 wants to be close to the helados (ice cream) cart. The helados cart is at the pier with probability 1/2 and at the lifeguard station with probability 1/2. In other words, if the cart is at the pier, her utility function is

\[ u_1(\text{pier}, a_1, a_2, a_3) = -(a_1 - 0)^2 - (a_1 - a_2)^2 - (a_1 - a_3)^2. \]

If the cart is at the lifeguard station, her utility function is

\[ u_1(\text{lifeguard}, a_1, a_2, a_3) = -(a_1 - 1)^2 - (a_1 - a_2)^2 - (a_1 - a_3)^2. \]

The only uncertainty is where the cart is. However, no one knows where the cart is. Person 2 and person 3 do not care about helados and thus their utility functions are the same as before:

\[ u_2(a_1, a_2, a_3) = -(a_2 - 1)^2 - (a_2 - a_1)^2 - (a_2 - a_3)^2 \]
\[ u_3(a_1, a_2, a_3) = -(a_3 - 1)^2 - (a_3 - a_1)^2 - (a_3 - a_2)^2 \]

Find the Nash equilibrium of this game. (3 points)
d. As before, the helados cart is at the pier with probability 1/2 and at the lifeguard station with probability 1/2. Thus we have

\[ u_1(\text{pier}, a_1, a_2, a_3) = -(a_1 - 0)^2 - (a_1 - a_2)^2 - (a_1 - a_3)^2. \]

\[ u_1(\text{lifeguard}, a_1, a_2, a_3) = -(a_1 - 1)^2 - (a_1 - a_2)^2 - (a_1 - a_3)^2. \]

Now say that Person 1 knows where the cart is. Persons 2 and 3 still do not know where the cart is. Person 2 and person 3 do not care about helados and thus their utility functions are the same as before:

\[ u_2(a_1, a_2, a_3) = -(a_2 - 1)^2 - (a_2 - a_1)^2 - (a_2 - a_3)^2 \]

\[ u_3(a_1, a_2, a_3) = -(a_3 - 1)^2 - (a_3 - a_1)^2 - (a_3 - a_2)^2 \]

Find the Nash equilibrium of this game. (3 points)