1. Say that a seller tries to sell a car to a buyer. The car is worth $1000 to the seller and $2000 to the buyer, and both people know this. First, the seller proposes a price of either $1800 or $1200 to the buyer. Given this price, the buyer can either accept or reject the offer. Model this as an extensive form game.

a. Find all Nash equilibria.
The easiest way to find all Nash equilibria of this game is to represent it as a strategic form game:

\[
\begin{array}{c|ccc}
$1800$ & $800, 200$ & $800, 200$ & $0, 0$ \\
$1200$ & $200, 800$ & $0, 0$ & $200, 800$ \\
\end{array}
\]

where the seller chooses the row and the buyer chooses the column. Then it is clear that the three (pure strategy) Nash equilibria are ($1800, if $1800, accept; if $1200, accept$), ($1800, if $1800, accept; if $1200, reject$), and ($1200, if $1800, reject; if $1200, accept$). Note that there is a Nash equilibrium where the buyer “holds out” for a price of $1200 and succeeds.

b. Which Nash equilibria are subgame perfect?
We note that in a subgame perfect Nash equilibrium, the buyer will accept either offer. So the only subgame perfect Nash equilibrium is ($1800, if $1800, accept; if $1200, accept$).

By backward induction, we find the following subgame Nash equilibrium:
2. Say that you and your brother divide up 4 oranges in the following manner. First, you divide the oranges up into two piles: the left pile and the right pile. For example, you could put one orange in the left pile and three in the right pile. Then, your brother decides which of the two piles he wants; he keeps that pile and you get the other pile. Both of you like oranges.

a. Model this as an extensive form game and find the subgame perfect Nash equilibria. How will the oranges be divided?

You have five possible strategies: put 4 oranges in the left pile and 0 in the right, 3 in the left and 1 in the right, 2 in the left and 2 in the right, 1 in the left and 3 in the right, and 0 in the left and 4 in the right (I abbreviate these as 4L, 0R, etc.) Your brother can choose in each of these contingencies whether to choose the left or right pile. We can say that payoffs are just the number of oranges one gets. Hence the extensive form game looks like this:

There are two subgame perfect Nash equilibria. The first, written as arrows in a game tree, is:
The second, written as arrows in a game tree, is:
If we were to spell these out explicitly, in the first equilibrium, person 1 plays strategy “2L, 2R” and person 2 plays strategy “If 4L, 0R, choose L; if 3L, 1R, choose L; if 2L, 2R, choose L; if 1L, 3R, choose R; if 0L, 4R, choose R”. The second equilibrium is similar. In both of these equilibria, you and your brother split the 4 oranges equally. Note that your brother’s strategy is the choice of either the left or right pile conditional on your division.

3. Say that country A can either attack or not attack country B. Country B cannot defend itself; the only thing it can do is to set off a “doomsday device” which obliterates both countries. The worst thing for both countries is to be obliterated; for country A, the best thing is to attack and not be obliterated; for country B, the best thing is to not be attacked.

a. Think of country A moving first and model this as an extensive form game. Find all pure strategy Nash equilibria and find the subgame perfect Nash equilibria. The game in extensive form looks like this (of course, your choice of payoffs might be different from mine):

```
A    B
attack  B
  |obliterate       |don't
  -10,-10          |5,-5
  don't

A    B
  |obliterate       |don't
  -10,-10          |0,0
  don't
```

The strategic form looks like this:

<table>
<thead>
<tr>
<th></th>
<th>if attack, obliterate; if attack, don’t; if not, obliterate</th>
<th>if attack, don’t; if not, don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>attack</td>
<td>-10,-10</td>
<td>5,-5</td>
</tr>
<tr>
<td>don’t</td>
<td>-10,-10</td>
<td>0,0</td>
</tr>
</tbody>
</table>

There are three pure strategy Nash equilibria: (don’t, if attack, obliterate; if not, don’t), (attack, if attack, don’t; if not, obliterate), (attack, if attack, don’t; if not, don’t). By backwards induction or whatever other method, the subgame perfect Nash equilibrium is (attack, if attack, don’t; if not, don’t).

b. Now think of country B moving first and model this as an extensive form game. Find all pure strategy Nash equilibria and find the subgame perfect Nash equilibria. The game in extensive form looks like this (even though now B moves first, I write country A’s payoffs first and country B’s payoffs second, so to remain consistent with part a. above).
The strategic form looks like this:

<table>
<thead>
<tr>
<th></th>
<th>obliterate</th>
<th>don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>if obliterated, attack; if not, attack</td>
<td>-10,-10</td>
<td>5,-5</td>
</tr>
<tr>
<td>if obliterated, attack; if not, don’t</td>
<td>-10,-10</td>
<td>0,0</td>
</tr>
<tr>
<td>if obliterated, don’t; if not, attack</td>
<td>-10,-10</td>
<td>5,-5</td>
</tr>
<tr>
<td>if obliterated, don’t; if not, don’t</td>
<td>-10,-10</td>
<td>0,0</td>
</tr>
</tbody>
</table>

The two Nash equilibria are (if obliterated, attack; if not, attack, don’t) and (if obliterated, don’t; if not, attack, don’t). These are both subgame perfect Nash equilibria.

4. The game called “Nim” goes like this: there are a pile of five stones on the ground. Player 1 can take either 1 or 2 stones. Then player 2 can take either 1 or 2 stones. They continue taking either 1 or 2 stones in turn until all the stones are gone. The player who takes the last stone (or stones) wins.

a. Model this as an extensive form game and find the subgame perfect Nash equilibria. The extensive form game is shown below.
Arrows in the game tree show one of the subgame perfect Nash equilibria. There are actually four subgame perfect Nash equilibria, because there are ties at the starred nodes. I won’t actually spell out all of the four subgame perfect Nash equilibria, since it would be tedious. In all of the subgame perfect Nash equilibria, player 1 initially takes 2 stones, and then no matter what player 2 then does, player 1 will eventually win.

b. What happens when you start with \( n \) stones instead of five?
You can show that if \( n \) is a multiple of 3, then in any subgame perfect Nash equilibrium, player 2 wins. Player 2 plays the following strategy: if player 1 picks one stone, then player 2 picks two stones; if player 1 picks two stones, then player 2 picks one stone. This way player 2 ensures that the remaining stones left after she plays is always a multiple of three.
If \( n \) is not a multiple of 3, then in any subgame perfect Nash equilibrium, player 1 wins. Player 1 initially picks up whatever number of stones which makes the remaining number of stones a multiple of three. From then on, if player 2 picks two stones, player 1 picks one stone, and if player 2 picks one stone, player 1 picks two stones.

5. Say we have a version of the “centipede” game:

\[
\begin{array}{cccc}
1 & \rightarrow & a & \rightarrow \\
| & & | & \\
1 & \rightarrow & a & \rightarrow \\
| & & | & \\
d & & d & \\
\downarrow & & \downarrow & \downarrow & \downarrow & \\
2, 2 & 1, 8 & 4, 4 & 2, 16
\end{array}
\]

a. Find the subgame perfect Nash equilibrium.

We can write down the subgame perfect equilibrium in the game tree below:

\[
\begin{array}{cccc}
1 & \rightarrow & a & \rightarrow \\
| & & | & \\
1 & \rightarrow & a & \rightarrow \\
| & & | & \\
d & & d & \\
\downarrow & & \downarrow & \downarrow & \downarrow & \\
2, 2 & 1, 8 & 4, 4 & 2, 16
\end{array}
\]

At each “node,” each player plays down. Person 1’s strategy is “First play down; if person 2 played across, play down.” Person 2’s strategy is “If person 1 played across, play down; if person 1 plays across again, then play down.”

b. Represent this game in strategic (matrix) form.

Remember that a strategy is a complete contingent plan. Person 1 can move either across or down, and in the contingency that person 2 moves across, can move either across or down. So we will represent his strategies as \( aa, ad, da, \) and \( dd \). If Person 1 moves across, Person 2 can move either across or down, and in the contingency that person 1 moves across again, can move either across or down. So we will represent his strategies also as \( aa, ad, da, \) and \( dd \). So the strategic form game looks like this:

\[
\begin{array}{cccc}
aa & ad & da & dd \\
8, 8 & 4, 4 & 2, 2 & 2, 2 \\
2, 16 & 1, 8 & 1, 8 & 1, 8 \\
1, 8 & 1, 8 & 1, 8 & 1, 8 \\
2, 2 & 2, 2 & 2, 2 & 2, 2 \\
2, 2 & 2, 2 & 2, 2 & 2, 2
\end{array}
\]

c. Solve this strategic form game using iterated elimination of weakly dominated strategies. Show the order of elimination.

By inspection, we can see that for person 2, \( dd \) and \( ad \) weakly dominate \( aa \). So we are left with
Now for person 1, both $da$ and $dd$ weakly dominate $aa$. So we are left with

\[
\begin{array}{ccc}
    ad & da & dd \\
    aa & 2,16 & 1,8 & 1,8 \\
    ad & 4,4 & 1,8 & 1,8 \\
    da & 2,2 & 2,2 & 2,2 \\
    dd & 2,2 & 2,2 & 2,2 \\
\end{array}
\]

Now it is clear that for person 2, both $da$ and $dd$ weakly dominate $ad$. Hence we are left with

\[
\begin{array}{cc}
    da & dd \\
    ad & 1,8 & 1,8 \\
    da & 2,2 & 2,2 \\
    dd & 2,2 & 2,2 \\
\end{array}
\]

Now for person 1, both $da$ and $dd$ weakly dominate $ad$. So finally, we have

\[
\begin{array}{cc}
    da & dd \\
    da & 2,2 & 2,2 \\
    dd & 2,2 & 2,2 \\
\end{array}
\]

It is clear that there can be no more domination. So the prediction is that person 1 will play either $da$ or $dd$, and person 2 will play either $da$ or $dd$. In any of these cases, the outcome will be that they both get a payoff of 2.

6. [from Spring 2004 midterm] Boyfriend and Girlfriend are on a 7 day cruise, which starts on Monday and goes until Sunday. Both know that this cruise is their last hurrah. Both know that once Sunday comes around and the cruise ends, if they are still together, the game ends because Boyfriend will be forced to dump Girlfriend (because otherwise he would be disowned by his wealthy parents).

On Monday evening, Boyfriend decides whether to break up or continue in the relationship. If Boyfriend decides to continue, then on Tuesday evening, Girlfriend decides whether to break up or to continue. If Girlfriend decides to continue, then on Wednesday evening, Boyfriend decides whether to continue. The game continues like this, with Boyfriend and Girlfriend taking turns deciding whether to continue or to break up (Boyfriend gets to decide on Monday, Wednesday, and Friday, and Girlfriend decides on Tuesday, Thursday, and Saturday). If anyone decides to break up, the game is immediately over. If no one decides to break up, then they make it to Sunday, and Girlfriend is left heartbroken.

Each person enjoys the others company, and gets a payoff of 5 for every day the relationship continues. However, no person wants to be dumped; if a person breaks up the relationship, the other person (the dumped one) gets 10 added to her payoff. For example, if they make it all the way to Sunday, Boyfriend’s payoff is 35 and Girlfriend’s payoff is 25.

a. Represent this as an extensive form game.
It’s just like the centipede game above. Note that on Sunday, Boyfriend has only one move. One could have written it so that Girlfriend has the last move, on Saturday.

b. Find the subgame perfect Nash equilibrium.
The subgame perfect Nash equilibrium is shown by the arrows above. Even though they enjoy each other’s company, Boyfriend and Girlfriend want to dump before being dumped. Hence Boyfriend dumps immediately, which is a sad thing for all.

7. Consider the “cross-out game.” In this game, one writes down the numbers 1, 2, 3. Person 1 starts by crossing out any one number or any two adjacent numbers: for example, person 1 might cross out 1, might cross out 1 and 2, or might cross out 2 and 3. Then person 2 also crosses out either one number or two adjacent numbers. For example, starting from 1, 2, 3, say person 1 crosses out 1. Then person 2 can either cross out 2, cross out 3, or cross out both 2 and 3. Play continues like this. Once a number is crossed out, it cannot be crossed out again. Also, if for example person 1 crosses out 2 in her first move, person 2 cannot then cross out both 1 and 3, because 1 and 3 are not adjacent (even though 2 is crossed out). The winner is the person who crosses out the last number.

a. Model this as an extensive form game. Show a subgame perfect Nash equilibrium of this game by drawing appropriate arrows in the game tree.
We did this in class.

b. Now instead of just three numbers, say that you start with \( m \) numbers. In other words, you have the numbers 1, 2, 3, \ldots, \( m \). Can person 1 always win this game? (Hint: look at \( m = 4, m = 5 \), etc. first to get some ideas.)
Person 1 can always win this game. His strategy is follows. If \( m \) is even, then person 1 starts by crossing out the two middle numbers. If \( m \) is odd, then person 1 starts out by crossing out the middle number. Henceforth, whatever number(s) person 2 crosses out, person 1 responds by crossing out the “mirror image” of those number(s) on the “other side” (if person 2 crosses out the number \( x \), person 1 crosses out \( m - x + 1 \)). Play continues like this, with each time person 1 responding in this manner to person 2. It is easy to see that person 1 is guaranteed to cross out the last number.
For example, say that \( m = 7 \). Person 1 crosses out 4, the middle number. If person 2 crosses out 1, then person 1 responds by crossing out 7. If person 2 then crosses out 5 and 6, then person 1 responds by crossing out 2 and 3.
Notice that this game is not related to the Nim game in the homework (there is no issue of multiples of 3, for example). Answers of the form “Person 1 will always win by crossing out the middle numbers” receive no credit without a full explanation like the one above.
8. Read “It’s Not Just About Bonds, It’s About Who’s on Deck” by Jack Curry on the “Media Clips” website. Model the situation using extensive form games and use the games to show that who bats behind Barry Bonds influences a pitcher’s decision of how to pitch to Bonds.

I model the situation in this way.

Here the Giants can either put in Alfonzo or Cruz behind Bonds. The Dodgers can either intentionally walk Bonds or not. If Alfonzo bats behind Bonds, and the Dodgers intentionally walk Bonds, putting him on base, they run the risk that Alfonzo will advance Bonds. But Alfonzo is hitting poorly enough these days that there is only a 30 percent chance that Bonds will advance. If the Dodgers do not intentionally walk Bonds and pitch to him, then Bonds will get on base with a 40 percent chance. If the Giants put in Cruz behind Bonds, then if the Dodgers intentionally walk Bonds, they would have to face Cruz, who is on a hot streak and will advance Bonds with a 50 percent chance. As before, if the Dodgers pitch to Bonds, then Bonds will get on base with a 40 percent chance.

The subgame perfect Nash equilibrium is shown in the game tree also. If the Giants put Alfonzo behind Bonds, then the Dodgers will walk Bonds, knowing that Alfonzo is slumping. If the Giants put Cruz behind Bonds, then the Dodgers will pitch to Bonds. Hence the Giants put Cruz behind Bonds.

9. Richard Nixon subscribed to a foreign-policy principle which he called the “madman theory” or the “theory of excessive force” (see “The Trials of Henry Kissinger” on the Media Clips website, or read “Nixon’s Nuclear Ploy” by William Burr and Jeffrey Kimball, also on the Media Clips website). We will explain this theory using an extensive form game.

Say that the US is facing an adversary (for example North Vietnam). The US makes the first move: it can either threaten North Vietnam or do nothing. If the US does nothing, then the game ends and “nothing happens.” If the US makes a threat, North Vietnam can either comply or not comply. If North Vietnam complies, then the game ends. If North Vietnam does not comply, then the US can either strike militarily or not.
a. Say the payoffs are like this: if the US does nothing, then both get payoff 0. If the US threatens and North Vietnam complies, then the US gets payoff 10 and North Vietnam gets payoff -5. If the US threatens, North Vietnam does not comply, and the US strikes, then the US gets payoff -10 and North Vietnam gets payoff -10. If the US threatens, North Vietnam does not comply, and the US does not strike, then the US gets payoff -5 and North Vietnam gets payoff 10.

The idea here is that the best thing for the US is for the US to make a threat and North Vietnam to comply; this way the US doesn’t have to perform the military strike. The worst thing for the US is to have to perform the military strike, since this would risk expanding the war. The best thing for North Vietnam is to openly defy the US and “call the bluff” of the US; the worst thing is the military strike. The second-worst thing is to comply to the US threat, and the second-best thing is to not be threatened at all.

Model this as an extensive form game and find the subgame perfect Nash equilibrium. The game looks like this.

The subgame perfect Nash equilibrium is shown with the arrows. Since a military strike is costly to both sides, the US will not strike if North Vietnam does not comply. Hence North Vietnam will not comply if threatened, and hence the US will not threaten. In a sense, if the US threatens, then North Vietnam will “call the bluff” because it knows that the US is not prepared to face the losses of a military strike.

b. Now say that Nixon, by purposefully showing the world that he is a “madman,” convinces everyone that the US payoff from a military strike is 5, not -10. In other words, Nixon tells the world that as far as he is concerned, he would enjoy a military strike. Model this as an extensive form game (all other payoffs are the same) and find the subgame perfect Nash equilibrium.

Now the game looks like this.
Now the US would love to engage in the military strike if North Vietnam does not comply. Hence North Vietnam complies, and hence the US can threaten.

c. Compare the two situations and explain why Nixon wants to show the world that he is a “madman.”
Nixon is better off in the second scenario. If people believe that the second scenario is the correct one (that Nixon is a “madman” in the sense that he wants to attack even if high casualties result) then Nixon can threaten and get his way without ever actually having to resort to military action.

10. Say we have two basketball players, Nash and Yao. Nash has the ball 20 feet from the basket and Yao is defending. Nash can either take a jump shot or drive the basket. Yao can either come out and contest the shot, or can stay in and protect the basket. The game looks like this.

\[
\begin{array}{c|cc}
  & Yao comes out & Yao protects \\
  \hline
  Nash takes jump shot & 0.3, 0.7 & 0.4, 0.6 \\
  Nash drives & 0.5, 0.5 & 0.3, 0.7 \\
\end{array}
\]

For example, if Nash takes a jump shot and Yao comes out, then Nash has a 30 percent chance of scoring and Yao has a 70 percent chance of successfully defending.

a. Find the mixed strategy Nash equilibrium of this game. In the mixed Nash equilibrium, what is Nash’s overall probability of scoring?
It’s easy to see that there are no pure strategy Nash equilibria. To find mixed strategy Nash equilibria, let’s set up our \( p \) and \( q \):

\[
\begin{bmatrix}
  [q] & [1-q] \\
  \hline
  [p] & \text{Nash takes jump shot} & 0.3, 0.7 & 0.4, 0.6 \\
  [1-p] & \text{Nash drives} & 0.5, 0.5 & 0.3, 0.7 \\
\end{bmatrix}
\]

First solve for \( q \). If Nash takes a jump shot, he gets payoff \( 0.3(q) + 0.4(1-q) \). If Nash drives, he gets payoff \( 0.5(q) + 0.3(1-q) \). To find the switchover value of \( q \), we set these equal and get \( 0.3(q) + 0.4(1-q) = 0.5(q) + 0.3(1-q) \), or in other words \( 0.4 - 0.1q = 0.3 + 0.2q \). Hence \( 0.1 = 0.3q \) and we have \( q = 1/3 \).
Now solve for $p$. If Yao comes out, he gets payoff $0.7p + 0.5(1 - p)$. If Yao protects, he gets $0.6p + 0.7(1 - p)$. To find the switchover value of $p$, we set these equal and get

$$0.7p + 0.5(1 - p) = 0.6p + 0.7(1 - p),$$

or in other words $0.5 + 0.2p = 0.7 - 0.1p$. Hence $0.3p = 0.2$ and we have $p = 2/3$.

So the mixed Nash equilibrium is that Nash takes a jump shot with probability 2/3 and drives with probability 1/3 and Yao comes out with probability 1/3 and protects with probability 2/3.

If Nash takes a jump shot, his overall probability of scoring is $0.3(1/3) + 0.4(2/3) = 1.1/3 = 0.37$. If Nash drives, his overall probability of scoring is $0.5(1/3) + 0.3(2/3) = 1.1/3 = 0.37$. Notice these numbers are the same because otherwise, Nash would not want to randomize. Hence Nash’s overall probability of scoring is 0.37.

b. Now say that Nash practices over the summer and improves his jump shot. Now the game looks like this.

<table>
<thead>
<tr>
<th></th>
<th>Yao comes out</th>
<th>Yao protects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nash takes jump shot</td>
<td>0.4, 0.6</td>
<td>0.5, 0.5</td>
</tr>
<tr>
<td>Nash drives</td>
<td>0.5, 0.5</td>
<td>0.3, 0.7</td>
</tr>
</tbody>
</table>

Find the mixed strategy Nash equilibrium of this new game. In the mixed Nash equilibrium, what is Nash’s overall probability of scoring?

Again, there are no pure strategy Nash equilibria. To find mixed strategy Nash equilibria, let’s set up our $p$ and $q$:

<table>
<thead>
<tr>
<th>$[q]$</th>
<th>$[1 - q]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[p]$</td>
<td></td>
</tr>
<tr>
<td>$[1 - p]$</td>
<td></td>
</tr>
</tbody>
</table>

Nash takes jump shot | 0.4, 0.6 | 0.5, 0.5 |
Nash drives           | 0.5, 0.5 | 0.3, 0.7 |

First solve for $q$. If Nash takes a jump shot, he gets payoff $0.4(q) + 0.5(1 - q)$. If Nash drives, he gets payoff $0.5(q) + 0.3(1 - q)$. To find the switchover value of $q$, we set these equal and get

$$0.4(q) + 0.5(1 - q) = 0.5(q) + 0.3(1 - q),$$

or in other words $0.5 - 0.1q = 0.3 + 0.2q$. Hence $0.2 = 0.3q$ and we have $q = 2/3$.

Now solve for $p$. If Yao comes out, he gets payoff $0.6p + 0.5(1 - p)$. If Yao protects, he gets $0.5p + 0.7(1 - p)$. To find the switchover value of $p$, we set these equal and get

$$0.6p + 0.5(1 - p) = 0.5p + 0.7(1 - p),$$

or in other words $0.5 + 0.1p = 0.7 - 0.2p$. Hence $0.3p = 0.2$ and we have $p = 2/3$.

So the mixed Nash equilibrium is that Nash takes a jump shot with probability 2/3 and drives with probability 1/3 and Yao comes out with probability 2/3 and protects with probability 1/3.

If Nash takes a jump shot, his overall probability of scoring is $0.4(2/3) + 0.5(1/3) = 1.3/3 = 0.43$. If Nash drives, his overall probability of scoring is $0.5(2/3) + 0.3(1/3) = 1.3/3 = 0.43$. Notice these numbers are the same because otherwise, Nash would not want to randomize. Hence Nash’s overall probability of scoring is 0.43.

c. After Nash improves his jump shot, does he go to his jump shot more often? After Nash improves his jump shot, is it more successful? After Nash improves his jump shot, is his drive more successful? Why does an improved jump shot improve other parts of Nash’s game?
When Nash improves his jump shot, he doesn’t go to it any more often; he still goes to it $1/3$ of the time. Nash’s jump shot improves from an overall 0.37 scoring probability to 0.43. Interestingly, his drive is more successful, also improving to an overall scoring probability of 0.43. His drive is more successful even though he hasn’t practiced it (he only practiced his jump shot). The reason his drive is more successful is because Yao has to come out more often (with probability $2/3$ instead of $1/3$) because of his improved jump shot. So it really is true that improving one aspect of your game improves the other aspects of your game, because it makes the defense react. Nash’s better jump shot makes Yao come out more, which improves the chances of Nash’s drive.