Corrected Answers to

Midterm exam  PS 30  November 2017

This is a closed book exam. The only thing you can take into this exam is yourself and writing instruments like pens and pencils. No calculators, computers, cell phones, headphones, etc. are allowed. Everything you write should be your own work. Cases of academic dishonesty will be referred to the Dean of Students office, which has the power to suspend and expel students. Partial credit will be given: math mistakes will not jeopardize your grade. Please show all steps of your work and explain what you are doing at each step. Correct answers alone are worth nothing without a clear and correct explanation of where the answers come from. Clarity and legibility are factors in the grade.

This exam has four problems. Each problem has 12 points. Please make sure that you have all four problems and that you complete or at least look at each one.

If you have a question, raise your hand and hold up the number of fingers which corresponds to the problem you have questions about (if you have a question on problem 2, hold up two fingers). If you need to leave the room during the exam (to use the restroom for example), you need to sign your name on the restroom log before leaving. You can only leave the room once.

When the end of the exam is announced, please stop working immediately. The exams of people who continue working after the end of the exam is announced will have their scores penalized by 30 percent. Please turn in your exam to your TA. When you hand in your exam, please write your name down on the log. Please write all answers on this exam—if you write on the reverse side of pages, please indicate this clearly. Good luck!

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Part 1. Say person 1 can either play $U$ or $D$. Person 2 can either play $L$ or $R$. They choose simultaneously. Their payoffs are given by the game below.

$$
\begin{array}{cc}
   & L & R \\
U & 0,0 & 4,4 \\
D & 1,7 & 2,1 \\
\end{array}
$$

a. Find all pure strategy Nash equilibria of this game. (2 points)

Pure strategy NE: $(U, R)$

$(D, L)$

b. Find all mixed strategy Nash equilibria of this game. (4 points)

$$
\begin{align*}
\bar{u}_1(U) &= 0.2 + 4(1-\rho) = 4-4\rho \\
\bar{u}_1(D) &= 1.5 + 2(1-\rho) = 2+2-2\rho \\
\bar{u}_2(L) &= 0.9 + 7(1-\rho) = 7-7\rho \\
\bar{u}_2(R) &= 4 - \rho + 1(1-\rho) = 4 - 2 \rho + 1 - \rho \\
\end{align*}
$$

$$
\begin{align*}
4-4\rho &= 2+2-2\rho \\
2 &= 3\rho \\
\rho &= \frac{2}{3} \\
7-7\rho &= 4\rho + 1-\rho \\
6 &= 10\rho \\
\rho &= \frac{6}{10} = \frac{3}{5} \\
\end{align*}
$$

Mixed NE:

$$
\left( \begin{array}{cc}
   1 & 0 \\
   0 & 2 \\
\end{array} \right)
\text{ with prob. } \frac{3}{5} \text{ and } \frac{2}{5},
\left( \begin{array}{cc}
   0 & 1 \\
   2 & 0 \\
\end{array} \right)
\text{ with prob. } \frac{2}{3} \text{ and } \frac{1}{3}
$$
c. Here is the game again.

\[
\begin{array}{cc}
L & R \\
U & 0, 0 & 4, 4 \\
D & 1, 7 & 2, 1
\end{array}
\]

Now say that they do not move simultaneously. They move in sequence. First person 1 chooses between \( U \) and \( D \). Then person 2, seeing person 1’s move, chooses between \( L \) and \( R \). Write this as an extensive form game. (2 points)

\[
\begin{array}{c}
2 \\
1 \\
\bigcup \\
\bigcap
\end{array}
\begin{array}{c}
\bigcup \\
\bigcap
\end{array}
\begin{array}{c}
L \quad \bigcup \quad 0, 0 \\
R \quad \bigcap \quad 4, 4
\end{array}
\begin{array}{c}
L \quad \bigcup \quad 1, 7 \\
R \quad \bigcap \quad 2, 1
\end{array}
\]

d. Find all pure strategy Nash equilibria of this extensive form game. (2 points)

\[
\begin{array}{cccc}
\bigcap & \bigcup & \bigcup & \bigcap \\
L & L & R & R \\
U & 0, 0 & 0, 0 & (4, 4) & (4, 4) \\
D & (1, 1) & 2, 1 & 1, 1 & 2, 1
\end{array}
\]

\( \text{NE: } (D, L) \quad (U, R) \quad (U, L) \quad (U, R) \)

e. Among the pure strategy Nash equilibria of this extensive form game, which are subgame perfect? (2 points)

\[
\begin{array}{c}
2 \\
1 \\
\bigcup \\
\bigcap
\end{array}
\begin{array}{c}
\bigcup \\
\bigcap
\end{array}
\begin{array}{c}
\bigcup \\
\bigcap
\end{array}
\begin{array}{c}
L \quad \bigcup \quad 0, 0 \\
R \quad \bigcap \quad 4, 4
\end{array}
\begin{array}{c}
L \quad \bigcup \quad 1, 7 \\
R \quad \bigcap \quad 2, 1
\end{array}
\]

\( (U, R) \text{ is the SPNE} \)
Part 2. A company has three partners, Katya, Linh, and Malala. They are choosing whether to invest in an amusement park (A), a bar (B), or a casino (C). They decide by voting. Since Katya is senior partner, Katya has three votes. Linh has two votes and Malala has two votes. The partners vote simultaneously, and the alternative with the most votes wins. For example, if Katya votes for A, Linh votes for B, and Malala votes for C, then A gets three votes, B gets two votes, and C gets two votes, and hence A wins.

Katya likes A best, B second-best, and C least. Linh likes B best, C second-best, and A least. Malala likes C best, A second-best, and B least. If a person gets her best choice, she gets payoff 2. If she gets her second-best, she gets payoff 1. If she gets her worst choice, she gets payoff 0.

Assume that a person will never vote for her last choice. So Katya votes either for A or B. Linh votes for either B or C. Malala votes for either C or A.

a. Write this as a strategic form game. (2 points)

b. Make a prediction in this game using iterative elimination of weakly and strongly dominated strategies. Try to eliminate as much as possible. Please write down your order of elimination. (2 points)

1. For Katya, A w. dom B.
2. For Malala, C w. dom A.
3. For Linh, C w. dom B.

Prediction: Katya plays A, Linh plays C, Malala plays C.

Other orders are possible.

c. Find all pure strategy Nash equilibria of this game. (1 point)

NE. (A, C, C) (A, B, A)

d. Katya has the most votes. Will Katya always get Katya’s preferred choice? If yes, please explain why. If not, please explain why not. (1 point)

No, it is possible that Katya will get Katya’s last choice, because (A, C, C) is a NE.
e. Now say that Linh becomes senior executive partner and now has four votes. Katya still has three votes and Malala has two votes. Write this as a strategic form game. (2 points)

\[
\begin{array}{c|ccc}
\text{Who wins:} & A & B & C \\
A & \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} & \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \\
B & \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} & \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \\
C & \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array} & \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}
\end{array}
\]

f. Make a prediction in this game using iterative elimination of weakly and strongly dominated strategies. Try to eliminate as much as possible. Please write down your order of elimination. (2 points)

1. For Linh, B w. dan C.
2. For Katya, A w. dan B
3. For Malala, A w. dan C.

Prediction: Katya plays A, Linh plays B, Malala plays A.

Other orders are possible.

\[
\text{NE}: (B, B, C), (A, B, A)
\]

h. Now Linh has the most votes. Will Linh always get Linh’s preferred choice? If yes, please explain why. If not, please explain why not. (1 point)

No, in fact Linh might get his worst choice, since \((A, B, A)\) is a NE.
Part 3. Aziz and Bao are thinking about going to the disco tonight. However, the disco closes at midnight and everyone has to leave when it closes. Aziz can either show up to the disco at 11:05pm, 11:10pm, or 11:15pm. Bao similarly can either show up at 11:05pm, 11:10pm, or 11:15pm. If they show up at exactly the same time, they bump into each other on the way in, which is kind of awkward, and they both get payoff $-100$. If they show up at different times, then their payoffs are the number of minutes they spend there together. For example, if Aziz shows up at 11:10pm and Bao shows up at 11:15pm, they are both there together for 45 minutes, and each gets payoff 45.

a. Write this as a strategic form game and find all pure strategy Nash equilibria. (3 points)

$$
\begin{array}{ccc}
11:05 & 11:10 & 11:15 \\
11:05 & -100,-100 & (50, 50) & 45, 45 \\
11:10 & (50, 50) & -100,-100 & 45, 45 \\
11:15 & 45, 45 & 45, 45 & -100,-100
\end{array}
$$

NE: $(11:10, 11:05)$, $(11:05, 11:10)$

b. Now say that Aziz is wearing a great new outfit and wants to make an entrance so that Bao can see it. The payoffs are the same as before, but now Aziz gets an additional bonus of 10 if Aziz arrives after Bao (this way Bao will be there when Aziz enters the disco). Bao’s payoffs are the same as before. For example, if Aziz shows up at 11:10pm and Bao shows up at 11:05pm, then Aziz’s payoff is 60 (they are together for 50 minutes and Aziz gets the bonus) and Bao’s payoff is 50. If Aziz shows up at 11:05pm and Bao shows up at 11:10pm, they both get a payoff of 50.

Write this as a strategic form game and find all pure strategy Nash equilibria. (3 points)

$$
\begin{array}{ccc}
11:05 & 11:10 & 11:15 \\
11:05 & -100,-100 & 50, 50 & 45, 45 \\
11:10 & (60, 50) & -100,-100 & 45, 45 \\
11:15 & 55, 45 & (55, 45) & -100,-100
\end{array}
$$

NE: $(11:10, 11:05)$, $(11:15, 11:10)$
c. Now say that Aziz can show up at 11:05pm, 11:10pm, 11:15pm, 11:20pm, 11:25pm, 11:30pm, 11:35pm, 11:40pm, 11:45pm, 11:50pm, 11:55pm, or midnight. Similarly, Bao can show up at 11:05pm, 11:10pm, 11:15pm, 11:20pm, 11:25pm, 11:30pm, 11:35pm, 11:40pm, 11:45pm, 11:50pm, 11:55pm, or midnight. For example, if Aziz shows up at midnight and Bao shows up at 11:15pm, Aziz’s payoff is 10 (because they spend 0 minutes together and Aziz gets the bonus) and Bao’s payoff is 0.

Find all pure strategy Nash equilibria of this game. (Don’t write out the entire game, which is quite large! Just think about it.) (3 points)

Bao’s best response is to show up before Aziz, because Bao does not want to show up at the same time as Aziz and Bao spends less time with Aziz if Bao shows up after Aziz.

Aziz’s best response is to show up five minutes after Bao, because Aziz gets the bonus of 10 but only loses 5 minutes of time with Bao.

So the NE are when Aziz shows up five minutes after Bao:

(11:10, 11:15), (11:15, 11:20), (11:20, 11:25) ... (12:00, 11:55).

d. Now say that Bao also has a great new outfit and also gets a bonus of 10 if Bao arrives after Aziz. For example, if Aziz arrives at 11:20pm and Bao arrives at 11:35pm, Aziz gets payoff 25 (because they spend 25 minutes together) and Bao gets payoff 35 (because Bao gets the bonus). If Aziz arrives at 11:45pm and Bao arrives at 11:10pm, then Aziz gets payoff 25 (because Aziz gets the bonus) and Bao gets payoff 15.

As before, Aziz can show up at 11:05pm, 11:10pm, 11:15pm, 11:20pm, 11:25pm, 11:30pm, 11:35pm, 11:40pm, 11:45pm, 11:50pm, 11:55pm, or midnight. Similarly, Bao can show up at 11:05pm, 11:10pm, 11:15pm, 11:20pm, 11:25pm, 11:30pm, 11:35pm, 11:40pm, 11:45pm, 11:50pm, 11:55pm, or midnight.

Find all pure strategy Nash equilibria of this game. (Don’t write out the entire game, which is quite large! Just think about it.) (3 points)

Now each person’s best response is to show up 5 minutes after the other person (as explained above), if possible.

When the other person shows up at midnight, a person’s best response is any strategy other than midnight (midnight gives a payoff of -100 and any other strategy gives a payoff of 0).

So the NE are

(11:55, 12:00) (12:00, 11:55)
Part 4. A challenger (C) is deciding whether to challenge the incumbent candidate (I) in an election.

First the challenger decides whether to enter the race (enter) or not (not). If the challenger does not enter the race, the game is over. If the challenger enters the race, the incumbent decides whether to fight (f) or to negotiate (n). If the incumbent chooses to fight (f), the challenger can then decide to either fight back (fb) or propose a deal (d). If the incumbent chooses to negotiate (n), the challenger can reject (r) or accept the proposal (a).

The extensive form game is shown below and the payoffs are written as (Challenger’s Payoff, Incumbent’s Payoff).

\[
\begin{array}{c}
\text{C} \\
\text{not} \\
\text{enter} \\
0, 2 \\
\text{f} \\
I \\
\text{C} \\
\text{fb} \\
1, -2 \\
\text{d} \\
-1, 1 \\
\text{r} \\
2, -3 \\
\text{a} \\
1, -1 \\
\end{array}
\]

a. Write this as a strategic form game. (4 points)

\[
\begin{array}{cccc}
\text{C} & \text{fb} & \text{r} & \text{f} \\
\text{not} & 0, 2 & 0, 2 & \text{f} \\
\text{C} & 0, 2 & 0, 2 & \text{f} \\
\text{C} & 0, 2 & 0, 2 & \text{f} \\
\text{C} & 0, 2 & 0, 2 & \text{f} \\
\text{e} & 0, 2 & 0, 2 & \text{f} \\
\text{f} & 2, -3 & 2, -3 & \text{f} \\
\text{d} & 1, -1 & 1, -1 & \text{f} \\
\text{r} & 2, -3 & 2, -3 & \text{f} \\
\text{a} & 1, -1 & 1, -1 & \text{f} \\
\end{array}
\]

b. Find all pure strategy Nash equilibria of this game. (4 points)

\[\mathbf{N} = \{ (e, f) \} \]
Here is the game again.

\begin{center}
\begin{tikzpicture}
    \node (C) at (0,0) {C};
    \node (not) at (-2,-2) {not};
    \node (enter) at (2,-2) {enter};
    \node (0,2) at (-2,-4) {0, 2};
    \node (C) at (1,-4) {C};
    \node (fb) at (-1,-6) {fb};
    \node (d) at (0,-6) {d};
    \node (r) at (1,-6) {r};
    \node (a) at (2,-6) {a};
    \node (1,-2) at (-1,-8) {1, -2};
    \node (-1,1) at (1,-8) {-1, 1};
    \node (2,-3) at (2,-8) {2, -3};
    \node (1,-1) at (3,-8) {1, -1};
    \draw (C) -- (not);
    \draw (C) -- (enter);
    \draw (not) -- (0,2);
    \draw (enter) -- (C);
    \draw (C) -- (fb);
    \draw (C) -- (d);
    \draw (C) -- (r);
    \draw (C) -- (a);
\end{tikzpicture}
\end{center}

c. Find all subgame perfect Nash equilibria of this game. (4 points)

SPNE as shown above

\((c, fb, r, f)\)