This is a closed book exam. The only thing you can take into this exam is yourself and writing instruments. No calculators, computers, cell phones, etc. are allowed. Everything you write should be your own work. Cases of academic dishonesty will be referred to the Dean of Students office, which has the power to suspend and expel students. Partial credit will be given: math mistakes will not jeopardize your grade. This exam has five questions. Question 1 is worth 15 points, question 2 is worth 14 points, question 3 is worth 12 points, question 4 is worth 12 points, and question 5 is worth 12 points. Please show all steps of your work and explain what you are doing at each step. Correct answers alone are worth nothing without a clear and correct explanation of where the answers come from. Clarity and legibility are factors in the grade.

If you need to leave the room during the exam (to use the restroom for example), you need to sign your name on the restroom log before leaving. You can only leave the room once.

When the end of the exam is announced, please stop working immediately. The exams of people who continue working after the end of the exam is announced will have their scores penalized by 30 percent. When you hand in your exam, please write your name down on the log. Please write all answers on this exam—if you write on the reverse side of pages, please indicate this clearly. Good luck!
1. Say that two firms are competing in an oligopoly. Firm 1 chooses $a_1$ and Firm 2 chooses $a_2$, and utility functions are given as follows. Note that the utility functions depend on the value of $\theta$.

\[
\begin{align*}
    u_1(a_1, a_2) &= (\theta - a_1 - a_2)a_1 \\
    u_2(a_1, a_2) &= (\theta - a_1 - a_2)a_2.
\end{align*}
\]

a. Say that $\theta = 90$. Find the Nash equilibrium of this game. (3 points)
b. Here are the utility functions again for your reference.

\[ u_1(a_1, a_2) = (\theta - a_1 - a_2)a_1 \]
\[ u_2(a_1, a_2) = (\theta - a_1 - a_2)a_2. \]

Now say that demand is either low, medium, or high. Each of these three possibilities are equally likely. If demand is low, then \( \theta = 60 \). If demand is medium, then \( \theta = 90 \). If demand is high, then \( \theta = 120 \). Say that both Firm 1 and Firm 2 completely know whether demand is low, medium, or high; they are both completely informed. Find the Nash equilibrium of this game. (3 points)
c. Here are the utility functions again for your reference.

\[ u_1(a_1, a_2) = (\theta - a_1 - a_2)a_1 \]
\[ u_2(a_1, a_2) = (\theta - a_1 - a_2)a_2. \]

Again, demand is either low (\(\theta = 60\)), medium (\(\theta = 90\)), or high (\(\theta = 120\)), and each of these three possibilities are equally likely. However, now Firm 1 knows whether demand is low, medium, or high but Firm 2 does not know anything about demand. In other words, Firm 1 is completely informed and Firm 2 is completely uninformed. Find the Nash equilibrium of this game. (3 points)
d. Here are the utility functions again for your reference.

\[ u_1(a_1, a_2) = (\theta - a_1 - a_2)a_1 \]

\[ u_2(a_1, a_2) = (\theta - a_1 - a_2)a_2. \]

As before, Firm 1 is completely informed and Firm 2 is completely uninformed, and all three states are equally likely. Now when demand is low, we have \( \theta = 80 \) and when demand is medium, we have \( \theta = 90 \). When demand is high, we have \( \theta = \theta_h \). Can you find a value of \( \theta_h \) such that in the Nash equilibrium of this game, Firm 2 always chooses 30? If so, find it. If not, explain why not. (3 points)
e. Here are the utility functions again for your reference.

\[ u_1(a_1, a_2) = (\theta - a_1 - a_2)a_1 \]

\[ u_2(a_1, a_2) = (\theta - a_1 - a_2)a_2. \]

As before, Firm 1 is completely informed and Firm 2 is completely uninformed, and all three states are equally likely. Now when demand is low, we have \( \theta = 50 \) and when demand is medium, we have \( \theta = 100 \). When demand is high, we have \( \theta = \theta_h \). Can you find a value of \( \theta_h \) such that in the Nash equilibrium of this game, Firm 1 chooses 10 when demand is low? If so, find it. If not, explain why not. (3 points)
2. Say country 1 and country 2 are in a conflict and each must decide whether to use a military option \( m \) or the diplomatic option \( d \). Country 1 has been trying to develop laser rifles, which would give it a very large military advantage. Country 2 has been trying to place a spy in Country 1’s weapons labs. Therefore there is uncertainty about whether Country 1 has laser rifles \( l \) or does not have laser rifles \( n \), and there is uncertainty about whether Country 2 has successfully placed a spy \( s \) or not \( n \). So there are four states of the world:

\[
\begin{array}{cc}
ls & ln \\
ns & nn
\end{array}
\]

For example, \( ns \) is the state in which Country 1 does not have laser rifles and Country 2 has successfully placed a spy. Each state is equally likely.

If Country 1 has successfully developed laser rifles (states \( ls \) and \( ln \)), then payoffs are as follows.

\[
\begin{array}{cc}
m & d \\
4, -8 & 8, -20 \\
-20, -4 & 0, 0
\end{array}
\]

If Country 1 has not developed laser rifles (states \( ns \) and \( nn \)), then payoffs are as follows. Note that if the other country chooses the military option and you don’t, you suffer greatly.

\[
\begin{array}{cc}
m & d \\
-8, -8 & -4, -20 \\
-20, -4 & 0, 0
\end{array}
\]

a. Consider the “standard” scenario. Country 1 of course knows whether it has laser rifles or not, but does not know whether Country 2 has placed a spy or not. If Country 2 has placed a spy, then it knows whether Country 1 has laser rifles or not, but if Country 2 has not placed a spy, then it does not know.

Model this as a game with incomplete information and find all Bayesian Nash equilibria. (Note that when Country 1 has laser rifles, then \( m \) dominates \( d \). Note also that when Country 2 knows that Country 1 has laser rifles, then Country 2 will therefore surely play \( m \). This allows you to simplify matters a lot: you only have to consider (at most) two possible strategies for Country 1 and (at most) four possible strategies for Country 2.) (4 points)
b. Now consider the “self-busted spy” scenario. As in the standard scenario, if Country 2 has placed a spy, then it knows whether Country 1 has laser rifles or not, and if Country 2 has not placed a spy, then it does not know. Country 2’s information partition is the same as before.

However, say that when Country 2’s spy reports back that Country 1 has not developed laser rifles, Country 2 reveals the existence of its spy to Country 1. In other words, when Country 1 has not developed laser rifles, it knows whether or not Country 2 has placed a spy.

Model this as a game with incomplete information and find all Bayesian Nash equilibria. (Again, note that when Country 1 has laser rifles, then \( m \) dominates \( d \). Note also that when Country 2 knows that Country 1 has laser rifles, then Country 2 will therefore surely play \( m \). Hence you only have to consider (at most) four possible strategies for Country 1 and (at most) four possible strategies for Country 2.) (4 points)

Here are the payoffs again for your reference.

If Country 1 has successfully developed laser rifles (states \( ls \) and \( ln \):

\[
\begin{array}{c|cc}
& m & d \\
\hline
m & 4, -8 & 8, -20 \\
d & -20, -4 & 0, 0 \\
\end{array}
\]

If Country 1 has not developed laser rifles (states \( ns \) and \( nn \):

\[
\begin{array}{c|cc}
& m & d \\
\hline
m & -8, -8 & -4, -20 \\
d & -20, -4 & 0, 0 \\
\end{array}
\]
c. Now consider the “inspection regime” scenario. As in the standard scenario, Country 1 does not know whether Country 2 has placed a spy or not; all Country 1 knows is whether it has laser rifles or not. However, now Country 1 fully discloses its weapons program to Country 2. Now Country 2 knows whether or not Country 1 has laser rifles regardless of whether it placed a spy or not. Country 2 knows everything.

Model this as a game with incomplete information and find all Bayesian Nash equilibria. (Again, note that when Country 1 has laser rifles, then \( m \) dominates \( d \). Note also that when Country 2 knows that Country 1 has laser rifles, then Country 2 will therefore surely play \( m \). Hence you only have to consider (at most) two possible strategies for Country 1 and (at most) four possible strategies for Country 2.) (4 points)

Here are the payoffs again for your reference.

If Country 1 has successfully developed laser rifles (states \( ls \) and \( ln \)):

\[
\begin{array}{ccc}
m & d & \\
m & 4, -8 & 8, -20 \\
d & -20, -4 & 0, 0 \\
\end{array}
\]

If Country 1 has not developed laser rifles (states \( ns \) and \( nn \)):

\[
\begin{array}{ccc}
m & d & \\
m & -8, -8 & -4, -20 \\
d & -20, -4 & 0, 0 \\
\end{array}
\]

d. Which of these three scenarios is most favorable to peace and why? (2 points)
3. Consider the following game, in which Nature moves first.

Represent this game as a strategic form game and find all Nash equilibria. (3 points)
a. Find all Perfect Bayesian Nash equilibria of this game. I write it down several times so you don’t have to spend time writing the trees over and over again. (9 points)
4. Please note that the four games below are similar but are all slightly different.

a. Consider the game below, with beliefs written in. Can you argue that the beliefs do not make sense using a “refinement” argument? If so, please make the argument. If not, please explain why not. (3 points)

b. Consider the game below, with beliefs written in. Can you argue that the beliefs do not make sense using a “refinement” argument? If so, please make the argument. If not, please explain why not. (3 points)
c. Consider the game below, with beliefs written in. Can you argue that the beliefs do not make sense using a “refinement” argument? If so, please make the argument. If not, please explain why not. (3 points)

![Game Diagram](image)

d. Consider the game below, with beliefs written in. Can you argue that the beliefs do not make sense using a “refinement” argument? If so, please make the argument. If not, please explain why not. (3 points)

![Game Diagram](image)
5.

a. Say that person 1 and person 2 each choose a position on a beach. In other words, person 1 chooses $a_1$ from set $A_1 = [0, 1]$ and person 2 chooses $a_2$ from set $A_2 = [0, 1]$. Their utility functions are given by:

$$u_1(a_1, a_2) = -(a_1 - a_2)^2$$
$$u_2(a_1, a_2) = -(a_1 - a_2)^2 - (a_2)^2.$$ 
Find all (pure strategy) Nash equilibria of this game. (3 points)

b. Now say that their utility functions are given by:

$$u_1(a_1, a_2) = (a_1 - a_2)^2$$
$$u_2(a_1, a_2) = -(a_1 - a_2)^2 - (a_2)^2.$$ 
Find all (pure strategy) Nash equilibria of this game. (3 points)
c. Now say that their utility functions are given by:

\[ u_1(a_1, a_2) = |a_1 - a_2| \]
\[ u_2(a_1, a_2) = -(a_1 - a_2)^2 - (a_2)^2. \]

Here \(|\cdot|\) indicates the absolute value (for example, \(|5| = 5\) and \(|-2| = 2\)). Find all (pure strategy) Nash equilibria of this game. (3 points)

d. Now say that their utility functions are given by:

\[ u_1(a_1, a_2) = (a_1 - a_2)^2 \]
\[ u_2(a_1, a_2) = (a_1 - a_2)^2 - (a_2)^2. \]

Find all (pure strategy) Nash equilibria of this game. (3 points)