This is a closed book exam. The only thing you can take into this exam is yourself and writing instruments. No calculators, computers, cell phones, etc. are allowed. Everything you write should be your own work. Cases of academic dishonesty will be referred to the Dean of Students office, which has the power to suspend and expel students. Partial credit will be given: math mistakes will not jeopardize your grade. This exam has four questions. Question 1 is worth 12 points, question 2 is worth 14 points, question 3 is worth 12 points, and question 4 is worth 12 points. Please show all steps of your work and explain what you are doing at each step. Correct answers alone are worth nothing without a clear and correct explanation of where the answers come from. Clarity and legibility are factors in the grade.

If you need to leave the room during the exam (to use the restroom for example), you need to sign your name on the restroom log before leaving. You can only leave the room once.

When the end of the exam is announced, please stop working immediately. The exams of people who continue working after the end of the exam is announced will have their scores penalized by 30 percent. When you hand in your exam, please write your name down on the log. Please write all answers on this exam—if you write on the reverse side of pages, please indicate this clearly. Good luck!
1. Consider the two-person game below, in which Nature moves first.

a. Find all Nash equilibria of this game. (4 points)
b. Find all Perfect Bayesian Nash equilibria of this game. I write it down several times so you don’t have to spend time writing the trees over and over again. (8 points)
2. Say that a seller is trying to sell a used car to a buyer. The car is either bad (with probability 1/2) or good (with probability 1/2). A bad car is worth 3 to the buyer and 1 to the seller. A good car is worth 7 to the buyer and 5 to the seller. The game goes like this. First nature decides whether the car is bad or good. The seller then offers to sell the car at either the price of 4 or the price of 6. Given this offer, the buyer then decides whether to accept or reject the offer.

a. Say that everyone knows whether the car is bad or good. We then have a game with complete information which looks like this. Payoffs are written as (seller, buyer). Find a subgame perfect Nash equilibrium of the game. (You do not have to find all subgame perfect Nash equilibria. You can notate the subgame perfect Nash equilibrium using arrows in the tree.) (4 points)

\[
\text{Seller's payoff: } \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}
\]

\[
\text{Buyer's payoff: } \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}
\]
b. Now say that neither seller nor buyer knows whether the car is bad or good. The game now looks like this. Find all perfect Bayesian Nash equilibria of this game. I write it down several times on the next page so you don’t have to spend time writing the trees over and over again. (4 points)

\[
\begin{array}{c|c|c|c|c}
N & \text{bad} & \text{good} \\
\hline
S & 4 & 6 & 4 & 6 \\
B & a & r & a & r \\
& 3, -1 & 0, 0 & -1, 3 & 0, 0 \\
& 5, -3 & 0, 0 & 1, 1 & 0, 0 \\
& a & r & a & r \\
\end{array}
\]

\[
\begin{align*}
NE: & \quad (4, 4) \quad (2, 2) \\
& \quad (6, 6) \quad (1, 1)
\end{align*}
\]
\[
\begin{align*}
\text{bad} & \quad \text{good} \\
1/2 & \quad 1/2 \\
0 & \quad 0 \\
-1, 3 & \quad 0, 0 \\
\end{align*}
\]
c. Now say that the seller knows whether the car is bad or good but the buyer does not. The game now looks like this. Find all perfect Bayesian Nash equilibria of this game. I write it down several times on the next page so you don’t have to spend time writing the trees over and over again. (4 points)
d. Which of the previous scenarios, a., b., or c., is worst for the seller? Which of the previous scenarios, a., b., or c., is worst for the buyer? (2 points)

The worst scenario for the seller is c. Because in the only PBE, the seller gets payoff 0. In the other scenarios, the seller has a chance of getting a higher payoff.

Similarly, the worst scenario for the buyer is c. In other words, when the seller knows and the buyer does not, it hurts both seller and buyer.
3. Say that person 1 and person 2 are deciding how much to study for tomorrow’s exam. Person 1’s utility function is given by
\[ u_1(\theta, a_1, a_2) = (\theta + a_2 - a_1) a_1 \]
where \( \theta \) is how important the exam is for getting a job. Person 2’s utility function is
\[ u_2(\theta, a_1, a_2) = (\theta + a_1 - a_2) a_2 \].

a. Say that \( \theta \) is either 12 (low importance), 18 (medium importance), or 24 (high importance). Each of these are equally likely. Say that neither person 1 nor person 2 knows what \( \theta \) is. Find the Bayesian Nash equilibrium of this game. (3 points)

\[
\mathbb{E}u_1 = \frac{1}{3} (12 + a_2 - a_1) a_1 + \frac{1}{3} (18 + a_2 - a_1) a_1 + \frac{1}{3} (24 + a_2 - a_1) a_1
\]
\[
\mathbb{E}u_2 = \frac{1}{3} (12 + a_1 - a_2) a_2 + \frac{1}{3} (18 + a_1 - a_2) a_2 + \frac{1}{3} (24 + a_1 - a_2) a_2
\]

\[
\frac{d\mathbb{E}u_1}{da_1} = \frac{1}{3} (12 + a_2 - 2a_1) + \frac{1}{3} (18 + a_2 - 2a_1) + \frac{1}{3} (24 + a_2 - 2a_1) = 0
\]
\[
12 + a_2 - 2a_1 + 18 + a_2 - 2a_1 + 24 + a_2 - 2a_1 = 0
\]
\[
54 + 3a_2 = 6a_1
\]
\[
9 + a_2 = a_1
\]

\[
\frac{d\mathbb{E}u_2}{da_2} = \frac{1}{3} (12 + a_1 - 2a_2) + \frac{1}{3} (18 + a_1 - 2a_2) + \frac{1}{3} (24 + a_1 - 2a_2) = 0
\]
\[
12 + a_1 - 2a_2 + 18 + a_1 - 2a_2 + 24 + a_1 - 2a_2 = 0
\]
\[
54 + 3a_1 = 6a_2
\]
\[
9 + \frac{a_1}{2} = a_2
\]
\[
9 + \frac{1}{2} (9 + \frac{a_2}{2}) = a_2
\]
\[
18 + 9 + \frac{a_2}{2} = 2a_2
\]
\[
27 = \frac{3a_2}{2} \quad \Rightarrow \quad a_2 = \frac{2}{3} \cdot 27 = 18
\]
\[
a_1 = 9 + \frac{18}{2} = 18
\]

NE = \((18, 18, 18), (18, 18, 18)\)
b. Now say that person 1 knows exactly what $\theta$ is but person 2 does not know what $\theta$ is. Find the Bayesian Nash equilibrium of this game. (3 points)

\[ E_{u_1} = \frac{1}{3} (12 + a_2 - a_1) a_1^{l} + \frac{1}{3} (18 + a_2 - a_2) a_1^{m} + \frac{1}{3} (2a + a_2 - a_1) a_1^{h} \]

\[ E_{u_2} = \frac{1}{3} (12 + a_1 - a_2) a_2^{l} + \frac{1}{3} (18 + a_1 - a_2) a_2^{m} + \frac{1}{3} (2a + a_1 - a_2) a_2^{h} \]

\[ \frac{dE_{u_1}}{da_1^{l}} = \frac{1}{3} (12 + a_2 - 2a_1^{l}) = 0 \quad \frac{dE_{u_1}}{da_1^{m}} = \frac{1}{3} (18 + a_2 - 2a_1^{m}) = 0 \]

\[ 12 + a_2 = 2a_1^{l} \quad 18 + a_2 = 2a_1^{m} \]

\[ \frac{6 + a_2}{2} = a_1^{l} \quad a_1^{m} \]

\[ \frac{dE_{u_1}}{da_1^{h}} = \frac{1}{3} (2a + a_2 - 2a_1^{h}) = 0 \quad \frac{dE_{u_2}}{da_2} = \frac{1}{3} (12 + a_1 - 2a_2) = 0 \]

\[ 2a + a_2 = 2a_1^{h} \quad 12 + a_1 - 2a_2 = 0 \]

\[ 54 + a_1^{l} + a_1^{m} + a_1^{h} = 6a_2 \]

\[ 54 + 6 + \frac{a_2}{2} + 9 + \frac{a_2}{2} + 12 + \frac{a_2}{2} = 6a_2 \]

\[ 81 + \frac{3a_2}{2} = 6a_2 \]

\[ 162 + 3a_1 = 12a_2 \Rightarrow 162 = 7a_2 \Rightarrow a_2 = 18 \]

\[ a_1^{l} = 6 + \frac{18}{2} = 15 \quad a_1^{m} = 9 + \frac{18}{2} = 18 \quad a_1^{h} = 12 + \frac{18}{2} = 21 \]

\[ \text{NE: } (15, 18, 21) \]
c. Now say that person 1 knows exactly what $\theta$ is. Person 2 knows whether $\theta$ is high importance or not, but cannot tell if $\theta$ is of low or medium importance. Find the Bayesian Nash equilibrium of this game. (3 points)

$$\mathbb{E}u_1 = \frac{1}{3} (12 + a_2^L - a_1^L) a_1^L + \frac{1}{3} (18 + a_2^L - a_1^M) a_1^M + \frac{1}{3} (24 + a_2^L - a_1^H) a_1^H$$

$$\mathbb{E}u_2 = \frac{1}{3} (12 + a_1^L - a_2^L) a_2^L + \frac{1}{3} (18 + a_1^L - a_2^M) a_2^M + \frac{1}{3} (24 + a_1^L - a_2^H) a_2^H$$

$$\frac{\partial \mathbb{E}u_1}{\partial a_1^L} = \frac{1}{3} (12 + a_2^L - 2a_1^L) = 0$$

$$12 + a_2^L = 2a_1^L$$

$$\frac{\partial \mathbb{E}u_2}{\partial a_2^L} = \frac{1}{3} (18 + a_1^L - 2a_2^L) = 0$$

$$18 + a_1^L = 2a_2^L$$

$$\frac{\partial \mathbb{E}u_1}{\partial a_1^H} = \frac{1}{3} (24 + a_1^L - 2a_2^H) = 0$$

$$24 + a_1^L = 2a_2^H$$

$$\frac{\partial \mathbb{E}u_2}{\partial a_2^H} = \frac{1}{3} (24 + a_2^L - 2a_1^H) = 0$$

$$24 + a_2^L = 2a_1^H$$

$$\frac{\partial \mathbb{E}u_1}{\partial a_2^M} = \frac{1}{3} (12 + a_1^L - 2a_2^M) + \frac{1}{3} (18 + a_1^M - 2a_2^M) = 0$$

$$30 + a_1^L + a_1^M = 4a_2^M$$

$$\frac{15}{2} + \frac{a_1^L}{4} + \frac{a_1^M}{4} = a_2^M$$

$$\frac{\partial \mathbb{E}u_1}{\partial a_1^L} = \frac{1}{3} (24 + a_1^L - 2a_2^H) = 0$$

$$24 + a_1^L = 2a_2^H$$

$$\frac{15}{2} + \frac{1}{4} (6 + \frac{a_2^L}{2}) + \frac{1}{4} (9 + \frac{a_2^M}{2}) = a_2^L$$

$$30 + 6 + \frac{a_2^L}{2} + 9 + \frac{a_2^M}{2} = 4a_2^L$$

$$45 = 3a_2^L \Rightarrow a_2^L = 15$$

$$a_1^L = 6 + \frac{15}{2} = 13.5$$

$$a_1^M = 7 + \frac{15}{2} = 16.5$$

$$12 + \frac{1}{2} (12 + \frac{a_2^H}{2}) = a_2^H$$

$$24 + 12 + \frac{a_2^H}{2} = 2a_2^H$$

$$36 = \frac{3}{2} a_2^H \Rightarrow a_2^H = 24$$

$$\mathbf{N}\mathbf{E}: \left( \begin{array}{ccc} 13.5 & 16.5 & 24 \\ 15 & 15 & 24 \end{array} \right)$$
d. Now say that person 1 does not know anything about what $\theta$ is. Person 2 knows whether $\theta$ is high importance or not, but cannot tell if $\theta$ is of low or medium importance. Find the Bayesian Nash equilibrium of this game. (3 points)

\[
\mathbb{E}u_1 = \frac{1}{3} \left( 12 + a_{1,m} - a_1 \right) q_1 + \frac{1}{3} \left( 18 + a_{1,m} - a_1 \right) a_1 + \frac{1}{3} \left( 24 + a_{2}^H - a_1 \right) a_1
\]

\[
\mathbb{E}u_2 = \frac{1}{3} \left( 12 + a_{1,m} - a_2 \right) q_2 + \frac{1}{3} \left( 18 + a_{1,m} - a_2 \right) a_2 + \frac{1}{3} \left( 24 + a_{1} - a_2 \right) a_2
\]

\[
\frac{d\mathbb{E}u_1}{da_1} = \frac{1}{3} \left( 12 + a_{1,m} - 2a_1 \right) + \frac{1}{3} \left( 18 + a_{1,m} - 2a_1 \right) + \frac{1}{3} \left( 24 + a_{2}^H - 2a_1 \right) = 0
\]

\[
12 + a_{1,m} - 2a_1 + 18 + a_{1,m} - 2a_1 + 24 + a_{2}^H - 2a_1 = 0
\]

\[
54 + 2a_{1,m} + a_{2}^H = 6a_1
\]

\[
\frac{d\mathbb{E}u_2}{da_{2,m}} = \frac{1}{3} \left( 12 + a_{1} - 2a_{2,m} \right) + \frac{1}{3} \left( 18 + a_{1} - 2a_{2,m} \right) = 0
\]

\[
30 + 2a_1 = 4a_{2,m}
\]

\[
15 + a_1 = 2a_{2,m}
\]

\[
\frac{d\mathbb{E}u_2}{da_{2}^H} = \frac{1}{3} \left( 24 + a_{1} - 2a_{2}^H \right) = 0
\]

\[
24 + a_1 = 2a_{2}^H
\]

\[
12 + \frac{a_{1}}{2} = a_{2}^H
\]

\[
54 + 15 + a_1 + 12 + \frac{a_{1}}{2} = 6a_1
\]

\[
81 + 3a_1 = 6a_1 \Rightarrow 162 + 3a_1 = 12a_1 \Rightarrow 162 = 9a_1 \Rightarrow a_1 = 18
\]

\[
2a_{2,m} = 15 + 18 = 33 \Rightarrow a_{2,m} = 16.5
\]

\[
a_{2}^H = 12 + \frac{18}{2} = 21
\]

\[
\mathbf{NE} = \left( \begin{array}{ccc}
18 & 18 & 18 \\
16.5 & 16.5 & 21
\end{array} \right)
\]
4. Say that there are two friends who are each deciding where to locate at the beach. Person 1 chooses location $a_1$ and person 2 chooses location $a_2$, where $a_1, a_2 \in [0, 1]$. The pier is at position 0 and the lifeguard station is at position 1. Both people like to be near the helados (ice cream) cart. The helados cart is either at the pier or the lifeguard station. If the helados cart is at the pier, their utility functions are

$$u_1(\text{pier}, a_1, a_2) = -(a_1 - 0)^2 - (a_1 - a_2)^2$$

$$u_2(\text{pier}, a_1, a_2) = -(a_2 - 0)^2 - (a_1 - a_2)^2.$$ 

If the helados cart is at the lifeguard station, their utility functions are

$$u_1(\text{lifeguard}, a_1, a_2) = -(a_1 - 1)^2 - (a_1 - a_2)^2$$

$$u_2(\text{lifeguard}, a_1, a_2) = -(a_2 - 1)^2 - (a_1 - a_2)^2.$$

a. Say that the helados cart is at the pier with probability $\frac{2}{3}$ and the lifeguard station with probability $\frac{1}{3}$. Say that person 1 knows where the helados cart is but person 2 does not. Find the Bayesian Nash equilibrium of this game. (6 points)
b. Say that the helados cart is at the pier with probability \( p \) and the lifeguard station with probability \( 1 - p \), where \( p \in [0, 1] \). Say that person 1 knows where the helados cart is but person 2 does not. The Bayesian Nash equilibrium will depend on the value of \( p \). Find all values of \( p \) such that in the Bayesian Nash equilibrium, someone (either person 1 or person 2) sometimes chooses to locate at position 1/4. (6 points)

\[
E_{U_1} = p \left( -a_1^p - (a_1^p - a_2^p)^2 \right) + (1-p) \left( -a_1^L - (a_1^L - a_2^L)^2 \right)
\]

\[
= p \left( -a_1^p \right)^2 - (a_1^p - a_2^p)^2 + (1-p) \left( -a_1^L \right)^2 - (a_1^L - a_2^L)^2
\]

\[
\frac{dE_{U_1}}{da_1^p} = p \left( -2a_1^p - 2a_1^p + 2a_2^p \right) = 0
\]

\[
2a_2^p = 4a_1^p
\]

\[
a_1^p = \frac{a_2^p}{2}
\]

\[
E_{U_2} = p \left( -a_2^p - (a_2^p - a_1^p)^2 \right) + (1-p) \left( -a_2^L - (a_2^L - a_1^L)^2 \right)
\]

\[
= p \left( -a_2^p \right)^2 - (a_2^p - a_1^p)^2 + (1-p) \left( -a_2^L \right)^2 - (a_2^L - a_1^L)^2
\]

\[
\frac{dE_{U_2}}{da_2} = p \left( -2a_2 + 2a_1^p - 2a_2^p \right) + (1-p) \left( -2a_2 + 2a_1^L - 2a_2^L \right) = 0
\]

\[
p \left( -4a_2^p \right) + 2p a_1^p + (1-p) \left( -4a_2^L \right) + 2(1-p) a_1^L = 0
\]

\[
2p a_1^p + 2(1-p) + 2(1-p) a_1^L = 4a_2
\]

\[
2p \frac{a_1^p}{2} + 2(1-p) + 2(1-p) \left( \frac{1+a_2^p}{2} \right) = 4a_2
\]

\[
p a_2^p + 2(1-p) + (1-p) a_2 = 4a_2
\]

\[
p a_2^p + 2(1-p) + (1-p) a_2 = 4a_2
\]

\[
p a_2^p + 2(1-p) + (1-p) a_2 = 4a_2
\]

\[
3(1-p) + a_2 = 4a_2
\]

\[
a_2 = 1-p
\]

\[
a_1^p = \frac{1-p}{2} \quad \text{and} \quad a_1^L = \frac{1 + (1-p)}{2} = \frac{2-p}{2}
\]

So, \( \mathcal{N} = \left\{ \frac{1-p}{2}, \frac{2-p}{2}, (1-p) \right\} \)
\[ \text{NE: } \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{array} \right), \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

(Continue work here)

Person 2 locates at \( \frac{1}{4} \) when \( 1 - \rho = \frac{1}{4} \Rightarrow \rho = \frac{3}{4} \)

and the NE is \( \left( \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right) \)

\[ \begin{align*}
&= \left( \frac{1}{8}, \frac{3}{8}, \frac{1}{4} \right)
\end{align*} \]

Person 1 locates at \( \frac{1}{4} \) sometimes when

\[ \frac{1 - \rho}{2} = \frac{1}{4} \Rightarrow 1 - \rho = \frac{1}{2} \Rightarrow \rho = \frac{1}{2} . \]

Hence the NE is \( \left( \frac{1}{2}, \frac{3}{4}, \frac{1}{4} \right) \)

\[ \begin{align*}
&= \left( \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \right)
\end{align*} \]

Also, person 1 locates at \( \frac{1}{4} \) sometimes when

\[ \frac{2 - \rho}{2} = \frac{1}{4} \Rightarrow 2 - \rho = \frac{1}{2} \Rightarrow \rho = \frac{3}{2} \]

but this can’t happen because \( \rho \in [0,1] \).

So a person sometimes locates at \( \frac{1}{4} \) when

\[ \rho = \frac{3}{4} \quad \text{NE: } \left( \frac{1}{8}, \frac{3}{8}, \frac{1}{4} \right) \]

or \( \rho = \frac{1}{2} \quad \text{NE: } \left( \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \right) \)