Problems which you should be able to do easily

1. Consider the “Battle of the Sexes” game below.
\[
\begin{array}{cccc}
2a & 2b \\
1a & 2, 1 & 0, 0 \\
1b & 0, 0 & 1, 2 \\
\end{array}
\]
a. Find all Nash equilibria (pure strategy and mixed strategy) of this game.
The pure Nash equilibria are \((1a, 2a)\) and \((1b, 2b)\). To find the mixed Nash equilibria, let
person 1 play \(1a\) with probability \(p\) and \(1b\) with probability \(1 - p\). Let person 2 play \(2a\) with
probability \(q\) and \(2b\) with probability \(1 - q\).
Given person 2’s strategy, if person 1 plays \(1a\), he gets an expected payoff of \(2q + 0(1 - q) = 2q\).
If person 1 plays \(1b\), he gets an expected payoff of \(0q + 1(1 - q) = 1 - q\). To find person 1’s
“switchover” probability, we set \(2q = 1 - q\) and get \(q = 1/3\).
Given person 1’s strategy, if player 2 plays \(2a\), she gets an expected payoff of \(1p + 0(1 - p) = p\).
If person 2 plays \(2b\), she gets an expected payoff of \(0p + 2(1 - p) = 2 - 2p\). To find person
2’s “switchover” probability, we set \(p = 2 - 2p\) and get \(p = 2/3\).
Thus in the mixed strategy Nash equilibrium, person 1 plays \(1a\) with probability \(2/3\) and
plays \(1b\) with probability \(1/3\) and person 2 plays \(2a\) with probability \(1/3\) and \(2b\) with probability \(2/3\).
b. Are any strategies in this game weakly or strongly dominated?
No strategies are weakly or strongly dominated.

2. Consider the following game.
\[
\begin{array}{cccccc}
2a & 2b & 2c & 2d & 2e \\
1a & 63, -1 & 28, -1 & -2, 0 & -2, 45 & -3, 19 \\
1b & 32, 1 & 2, 2 & 2, 5 & 33, 0 & 2, 3 \\
1c & 54, 1 & 95, -1 & 0, 2 & 4, -1 & 0, 4 \\
1d & 1, -33 & -3, 43 & -1, 39 & 1, -12 & -1, 17 \\
1e & -22, 0 & 1, -13 & -1, 88 & -2, -57 & -3, 72 \\
\end{array}
\]
a. Find all pure strategy Nash equilibria of this game.
We can use our stars and pluses:
\[
\begin{array}{cccccc}
2a & 2b & 2c & 2d & 2e \\
1a & *63, -1 & 28, -1 & -2, 0 & -2, 45+ & -3, 19 \\
1b & 32, 1 & 2, 2 & *2, 5+ & *33, 0 & *2, 3 \\
1c & 54, 1 & *95, -1 & 0, 2+ & 4, -1 & 0, 4 \\
1d & 1, -33 & -3, 43+ & -1, 39 & 1, -12 & -1, 17 \\
1e & -22, 0 & 1, -13 & -1, 88+ & -2, -57 & -3, 72 \\
\end{array}
\]
So the only pure strategy Nash equilibrium is \((1b, 2c)\).
b. Make a prediction in this game by iteratively eliminating (strongly or weakly) dominated strategies.

First we note that 1c strongly dominates 1d and 1e.

\[
\begin{array}{c|ccccc}
   & 2a & 2b & 2c & 2d & 2e \\
1a & 63, -1 & 28, -1 & -2, 0 & -2, 45 & -3, 19 \\
1b & 32, 1 & 2, 2 & 2, 5 & 33, 0 & 2, 3 \\
1c & 54, 1 & 95, -1 & 0, 2 & 4, -1 & 0, 4 \\
\end{array}
\]

Next we note that 2c strongly dominates 2a and 2b.

\[
\begin{array}{c|cc}
   & 2d & 2e \\
1a & -2, 0 & -2, 45 & -3, 19 \\
1b & 2, 5 & 33, 0 & 2, 3 \\
1c & 0, 2 & 4, -1 & 0, 4 \\
\end{array}
\]

Next we note that 1b strongly dominates 1a and 1c.

\[
\begin{array}{c|cc}
   & 2d & 2e \\
1b & 2, 5 & 33, 0 & 2, 3 \\
\end{array}
\]

Finally, we note that 2c strongly dominates 2d and 2e.

\[
\begin{array}{c}
   2c \\
1b & 2, 5 \\
\end{array}
\]

So the prediction is that person 1 plays 1b and person 2 plays 2c.

3. [from Spring 2002 midterm] Mother can ask either Sister or Brother to do the dishes while she goes out shopping. If Mother asks Sister, she can either do it or not do it. If Mother asks Brother, he can either do it or not do it. Since Sister does a better job, Mother prefers Sister doing the dishes over Brother doing the dishes. However, Mother prefers Brother doing the dishes over them not being done.

Both Sister’s and Brother’s preferences are like this: the best thing is for the other person to do the dishes; the second best thing is for Mother to ask the other person and have the other person not do it (since then the other person will get blamed). The third best thing is to do the dishes, and the worst thing is to be asked to do the dishes but then not do it (since you will get in trouble).

The main thing here is to set up the players and strategies correctly. It is crucial that this is represented as a 3 person game.

<table>
<thead>
<tr>
<th></th>
<th>Do it</th>
<th>Don’t</th>
<th>Do it</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mom asks you</td>
<td>5, 3, 9</td>
<td>0, 0, 7</td>
<td>5, 3, 9</td>
<td>0, 0, 7</td>
</tr>
<tr>
<td>Mom asks brother</td>
<td>4, 9, 3</td>
<td>4, 9, 3</td>
<td>0, 7, 0</td>
<td>0, 7, 0</td>
</tr>
<tr>
<td>Brother does it</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brother doesn’t</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Find all (pure strategy) Nash equilibria.

By checking all the strategy profiles, we can see that (Mom asks you, You do it, Brother does it), (Mom asks you, You do it, Brother doesn’t), and (Mom asks brother, You don’t, Brother does it) are the three Nash equilibria.
Challenge problems

4. The simplest kind of game has two players, who each have two possible actions. We call these games “2 \times 2 games.”

a. Write down a 2 \times 2 game which has exactly one pure strategy Nash equilibrium and no mixed strategy Nash equilibrium. Solve for the equilibrium.
A Prisoners’ Dilemma works here.

\[
\begin{array}{cc}
2a & 2b \\
1a & 3,3 \ 0,4 \\
1b & 4,0 \ 1,1 \\
\end{array}
\]

There is only one pure strategy Nash equilibrium (1b, 2b) and no mixed strategy Nash equilibrium (since person 1 will never play 1a and person 2 will never play 2a).

b. Write down a 2 \times 2 game which has no pure strategy Nash equilibrium and exactly one mixed strategy Nash equilibrium. Solve for the equilibrium.
The penalty kick game works here.

\[
\begin{array}{cc}
2a & 2b \\
1a & 1,0 \ 0,1 \\
1b & 0,1 \ 1,0 \\
\end{array}
\]

As we did in class, in the mixed strategy Nash equilibrium, person 1 plays 1a with probability 1/2 and 1b with probability 1/2, and person 2 plays 2a with probability 1/2 and 2b with probability 1/2.

c. Write down a 2 \times 2 game which has exactly three total (pure and mixed) Nash equilibria. Solve for the equilibria.
The chicken game works here.

\[
\begin{array}{cc}
2a & 2b \\
1a & 3,3 \ 1,4 \\
1b & 4,1 \ 0,0 \\
\end{array}
\]

Here (1b, 2a) and (1a, 2b) are pure strategy Nash equilibria. If we compute the mixed strategy Nash equilibrium, we find that person 1 plays 1a with probability 1/2 and 1b with probability 1/2, and person 2 plays 2a with probability 1/2 and 2b with probability 1/2.

d. Write down a 2 \times 2 game in which the total number of (pure and mixed) Nash equilibria is neither one nor three. Solve for the equilibria.
The game in which “everything ties” works here.

\[
\begin{array}{cc}
2a & 2b \\
1a & 3,3 \ 3,3 \\
1b & 3,3 \ 3,3 \\
\end{array}
\]

Here there are four pure strategy Nash equilibria: (1a, 2a), (1a, 2b), (1b, 2a), (1b, 2b). In fact, if person 1 plays any mixed strategy and person 2 plays any mixed strategy, then that is a mixed strategy Nash equilibrium.
5. Say that country A and country I are at war. The two countries are separated by a system of rivers, as shown below.

```
A --- b --- c
    |     |     |
    d --- e --- f
    |     |     |
    g --- h --- I
```

Country I sends a naval fleet with just enough supplies to reach A. The fleet must stop for the night at intersections (for example, if the fleet takes the path IhebA, it must stop the first night at h, the second at e, and the third at b). If unhindered, on the fourth day the fleet will reach A and destroy country A. Country A can send a fleet to prevent this. Country A's fleet has enough supplies to visit three contiguous intersections, starting from A (for example Abcf). If it catches Country I's fleet (that is, if both countries stop for the night at the same intersection), it destroys the fleet and wins the war. Model this as a strategic form game, assuming that the winner gets payoff 1 and the loser gets payoff -1. Iteratively eliminate weakly dominated strategies and make some sort of prediction. Please see the attached page.
2.14 A Military Strategy Game

First we can eliminate all Country I strategies that don’t arrive at A. This leaves six strategies, which we can label fcb, fcb, fcb, hed, heb, and hgd. We can also eliminate all Country A strategies that stay at A at any time, or that hit h or f. This leaves the six strategies bcb, beb, bed, ded, deb, dgd. Here is the payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>bcb</th>
<th>beb</th>
<th>bed</th>
<th>ded</th>
<th>deb</th>
<th>dgd</th>
</tr>
</thead>
<tbody>
<tr>
<td>fcb</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>fcb</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>fed</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>hed</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>heb</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>hgd</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Now fcb is weakly dominated by fcb, as is heb. Moreover, we see that fed and hed are weakly dominated by hgd. Thus there are two remaining strategies for Country I, “south” (hgd) and “north” (fcb).

Also bcb is dominated by beb and dgd is dominated by ded, so we may drop them. Moreover, beb and deb are the same “patrol north”, while bed and ded are the same “patrol south.” This gives us the following reduced game:

<table>
<thead>
<tr>
<th></th>
<th>patrol north</th>
<th>patrol south</th>
</tr>
</thead>
<tbody>
<tr>
<td>attack north</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>attack south</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
</tbody>
</table>

So this complicated game is just the heads-tails game, which we will finish solving when we do mixed strategy equilibria!
6. [from Spring 2002 final] There are three people who each simultaneously choose the letter A or the letter B. If a person chooses a letter which no one else chooses, then he gets a payoff of 4; if a person chooses a letter which some other person chooses, then he gets a payoff of 0. The only exception is if everyone chooses the letter A, in which case everyone gets a payoff of 3.

a. Model this as a strategic form game and find all pure-strategy Nash equilibria. The game looks like this:

<table>
<thead>
<tr>
<th></th>
<th>2A</th>
<th>2B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>3, 3, 3</td>
<td>0, 4, 0</td>
</tr>
<tr>
<td>1B</td>
<td>4, 0, 0</td>
<td>0, 0, 4</td>
</tr>
<tr>
<td>3A</td>
<td>0, 0, 4</td>
<td>4, 0, 0</td>
</tr>
</tbody>
</table>

The pure strategy Nash equilibria are (1B, 2A, 3A), (1A, 2B, 3A), (1B, 2B, 3A), (1A, 2A, 3B), (1B, 2A, 3B), and (1A, 2B, 3B).

b. There exists a mixed-strategy Nash equilibrium in which each person plays A with probability \( p \) and B with probability \( 1 - p \). Find \( p \).

Say person 2 plays 2A with probability \( p \) and 2B with probability \( 1 - p \) and person 3 plays 3A with probability \( p \) and 3B with probability \( 1 - p \). If person 1 plays 1A, her expected payoff is \( 3p^2 + 4(1 - p)^2 \). If person 1 plays 1B, her expected payoff is \( 4p^2 \). To find the switchover probability, we set \( 3p^2 + 4(1 - p)^2 = 4p^2 \) and hence get \( 4(1 - p)^2 = p^2 \). Thus \( 4 - 8p + 4p^2 = p^2 \) and so \( 3p^2 - 8p + 4 = 0 \). Factoring, we get \( (3p - 2)(p - 2) = 0 \) and hence \( p = 2/3 \) (\( p = 2 \) is not possible because \( p \) is a probability). If you do the same thing with persons 2 and 3, you will get the same answer for \( p \) since the game is so symmetric. Hence in the mixed Nash equilibrium, each person plays A with probability 2/3 and B with probability 1/3.