Answers to

Midterm exam    PS 30    November 2018

This is a closed book exam. The only thing you can take into this exam is yourself and writing instruments like pens and pencils. No calculators, computers, cell phones, headphones, etc. are allowed. Everything you write should be your own work. Cases of academic dishonesty will be referred to the Dean of Students office, which has the power to suspend and expel students. Partial credit will be given: math mistakes will not jeopardize your grade. Please show all steps of your work and explain what you are doing at each step. Correct answers alone are worth nothing without a clear and correct explanation of where the answers come from. Clarity and legibility are factors in the grade.

This exam has four problems. Each problem has 12 points. Please make sure that you have all four problems and that you complete or at least look at each one.

If you have a question, raise your hand and hold up the number of fingers which corresponds to the problem you have questions about (if you have a question on problem 2, hold up two fingers). If you need to leave the room during the exam (to use the restroom for example), you need to sign your name on the restroom log before leaving. You can only leave the room once.

When the end of the exam is announced, please stop working immediately. The exams of people who continue working after the end of the exam is announced will have their scores penalized by 30 percent. Please turn in your exam to your TA. When you hand in your exam, please write your name down on the log. Please write all answers on this exam—if you write on the reverse side of pages, please indicate this clearly. Good luck!
Problem 1. Pete and Ariana have been dating for a month. Each person decides simultaneously whether to propose marriage or not. If they both propose, then they get engaged and each gets a payoff of 10. If only one person proposes, the person who proposes gets $-6$ (because of the embarrassment of making a failed proposal) and the person who does not gets 6 (because they will have an advantage over the other person when deciding on the future of their relationship). If no one proposes marriage, they continue to date and both get 0.

a. Model this as a strategic form game and find all pure strategy and mixed strategy Nash equilibria. (2 points)

\[
\begin{array}{c|cc}
\text{Pete} & \text{propose} & \text{not} \\
\hline
\text{propose} & (10, 10) & (-6, 6) \\
\text{not} & (6, -6) & (0, 0) \\
\end{array}
\]

**Pure strategy NE:** (propose, propose)  
(not, not)

\[
\begin{align*}
\text{EV Pete (propose)} &= 10\rho + (-6)(1-\rho) = -6 + 16\rho \\
\text{EV Pete (not)} &= 6\rho + 0(1-\rho) = 6\rho \\
\text{EV Ariana (propose)} &= 10\rho + (-6)(1-\rho) = -6 + 16\rho \\
\text{EV Ariana (not)} &= 6\rho + 0(1-\rho) = 6\rho \\
\end{align*}
\]

\[10\rho = 6 \implies \rho = \frac{3}{5}\]

\[-6 + 16\rho = 6\rho \implies -6 + 6\rho = 6\rho \implies -6 + 6\rho = 6\rho \implies \rho = \frac{3}{5}\]

**Mixed NE:**

\[
\left(\text{Pete proposes with prob } \frac{3}{5}, \text{ Ariana proposes with prob } \frac{3}{5}\right)
\]
b. Pete and Ariana get engaged, but soon afterward Ariana is convinced by her publicist to break off the engagement.

Now Ariana must decide whether to propose marriage again (and thereby stay in a relationship with Pete) or not. If Ariana proposes, Pete chooses to either accept or reject this proposal: if Pete accepts the proposal, the relationship survives, and if Pete rejects it, the relationship ends. If Ariana does not propose, Pete chooses whether to stay single or start dating someone else.

Ariana gets 15 if she proposes again and Pete accepts it. Ariana gets −10 if she proposes again and Pete rejects it. If Ariana does not propose again and Pete stays single, Ariana gets 0. However, if Ariana does not propose again and Pete starts dating someone else, Ariana gets −16 because he was the first to move on.

Pete gets 12 if he rejects Ariana’s new proposal because he gets payback for Ariana breaking off the first engagement. Pete gets 7 if he accepts the new proposal because he loses the chance to get payback. If Ariana does not propose again and Pete stays single, he gets 10. If Pete starts dating someone else, he gets 8.

Model this as an extensive form game and find a subgame perfect Nash equilibrium (if there is more than one, just write down one of them). (2 points)

c. Write this extensive form game as a strategic form game and find all pure strategy Nash equilibria of this game. (3 points)
d. Say Nicki, who is Ariana’s enemy, conspires to embarrass Ariana. Nicki changes the payoffs in the game in part b. above.

Nicki tries to motivate Ariana to propose again to Pete by offering benefits $x$ to Ariana: when Ariana proposes again and Pete rejects it, Ariana’s payoff is $-10 + x$. Nicki also promises benefits $y$ to Pete if he starts dating someone else: if Pete starts dating someone else, he gets payoff $8 + y$. All other payoffs are the same as in part b. above. Model this as an extensive form game. (2 points)

![Game Diagram]

e. The subgame perfect Nash equilibrium of this game depends on $x$ and $y$. Circle all the combinations of $x$ and $y$ below which ensure that in the resulting subgame perfect Nash equilibrium, Ariana proposes again and Pete rejects it. For example, if when $x = 4$ and $y = 1$, Ariana proposes again and Pete rejects it, circle $x=4$ below. Please explain your work. (3 points)

If $8 + y < 10$ ($y < 2$), then Pete stays single if Ariana does not propose.

So Ariana proposes if $-10 + x > 0$, in other words $x > 10$.

If $8 + y > 10$ ($y > 2$), then Pete dates someone else if Ariana does not propose.

So Ariana proposes if $-10 + x > -16$, in other words $x > 6$. 
Problem 2. Person 1 and Person 2 are playing a game. The word BRUIN is on the blackboard. Each person can either erase one letter from the right or two letters from the left. They take turns doing so and Person 1 goes first. For example, Person 1 can either erase the N, leaving BRUI, or erase BR, leaving UIN. If Person 1 erases the N, leaving BRUI, then Person 2 can either erase the BR, leaving UI, or erase the I, leaving BRU. If there is only one letter left, a person must erase that letter, leaving nothing. If there are just two letters left, then a person can either erase them both, leaving nothing, or erase the rightmost letter; for example, if the letters UI are on the board, then a person can erase them both and leave nothing or erase the I, with the U left on the board.

a. Say that U is the “poisoned letter”—whoever erases the U loses the game. Whenever anyone erases the U, the game immediately ends; the person who erased the U is the loser and the other person is the winner. Write this as an extensive form game. (2 points)

b. Write down a subgame perfect Nash equilibrium of this game (do not write down all of them). (1 point)

One of the SPNE is shown by the red arrows above.

C. How many subgame perfect Nash equilibria does this game have? (Do not write them all down; just write down how many there are and explain your answer.) (1 point)

Note that there are three nodes with ties (indicated by stars *). Each of these nodes has two actions which can be part of a SPNE. So there are $2 \times 2 \times 2 = 8$ SPNEs.
d. Again, we start with the word BRUIN on the blackboard. However, now I is the “poisoned letter”—whoever erases the I loses the game. Write this as an extensive form game. (2 points)

e. Write down a subgame perfect Nash equilibrium of this game (do not write down all of them). (1 point)

One of the SPNE is shown by the red arrows above.

f. How many subgame perfect Nash equilibria does this game have? (Do not write them all down; just write down how many there are and explain your answer.) (1 point)

There are three nodes where there are ties, indicated by stars *. So there are $2 \times 2 \times 2 = 8$ SPNEs.
g. Now say that the word ENCYCLOPEDIA is on the board. The “poisoned letter” is the P—whoever erases the P immediately loses. The rules of the game are the same as before. Can Person 1 guarantee a win in this game? Can Person 2 guarantee a win in this game? If so, explain why. If not, explain why not. (Do not write down an extensive form game, which would be complicated! Just think about it.) (4 points)

This game is much simpler than it appears. It actually isn’t much of a game. Since you never want to erase the P, there are only seven possible moves you can make without losing, as shown here:

\[
\begin{array}{cccc}
E & N & C & Y \\
C & L & O & P \\
P & E & D & I \\
A & & & \\
\end{array}
\]

For example, Person 1 can erase EN, then Person 2 erases CY, and so forth. Of course people can take these actions in different orders, but they all end up with Person 2 being left with OP, which makes Person 2 lose.

So Person 1 can guarantee a win.
Problem 3. Say that China is trying to secure a supply of lumber for construction. Thailand and Burma have lots of lumber, and China can either propose trade with Thailand or declare war on Burma. If China declares war on Burma, Burma can either resist or not. If China proposes trade with Thailand, Thailand can either accept China’s trade proposal or reject it. If Thailand rejects it, China can either impose tariffs on Thailand or not. We can represent the situation with the following game, where payoffs are given as (China, Thailand, Burma).

Note that the best thing for China is for Thailand to accept the trade deal. The best thing for Thailand is to reject China’s offer of trade and not have tariffs, because this enhances Thailand’s bargaining position. The worst thing for Burma (and China) is to fight a war.

a. Find a subgame perfect Nash equilibrium of this game (you do not have to write down all of them). (2 points)

\[ \text{SNE shown in red arrows above.} \]

\[ (\text{war not, reject, not}) \]
Here is the game again.

b. Represent this game as a strategic form game. (It is most convenient to let China be player 1, let Thailand be player 2, and let Burma be player 3; remember that payoffs are written as (China, Thailand, Burma).) (4 points)

c. Make a prediction in this game using iterative elimination of strongly and weakly dominated strategies. Try to eliminate as much as possible. Please indicate the order of elimination. (4 points)

1. For player 3, not war dominates resist.
2. For player 1, trade not war dominates trade tariffs.
3. For player 2, reject war dominates accept.
4. For player 1, war tariffs war dominates trade not.

(Note: other orders are possible)

d. Find all pure strategy Nash equilibria of this game. (2 points)

(trade tariffs, accept, resist) (trade tariffs, accept, not)
(trade not, reject, resist) (war tariffs, reject, not)
(war not, reject, not)
Problem 4. Arash, Bianca, and Ciaran are bringing oranges to a party. Arash can bring either 1 or 2 oranges. Bianca can bring either 3 or 4 oranges. Ciaran can bring either 5 or 6 oranges. They choose simultaneously. At the party, they put all the oranges together in one pile and enjoy eating them together. Arash wants there to be as many oranges as possible, and thus Arash’s payoff is the total number of oranges brought. Bianca likes oranges also but is superstitious about even numbers. So if the total number of oranges is even, Bianca’s payoff is 0; if the total number of oranges is odd, Bianca’s payoff is the total number of oranges. Ciaran is superstitious about odd numbers. If the total number of oranges is odd, Ciaran’s payoff is 0; if the total number of oranges is even, Ciaran’s payoff is the total number of oranges. For example, if Arash brings 1 orange, Bianca brings 3, and Ciaran brings 5, then Arash’s payoff is 9, Bianca’s is 9, and Ciaran’s is 0.

a. Write this as a strategic form game. (4 points)

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9,9,0</td>
<td>10,0,10</td>
</tr>
<tr>
<td>2</td>
<td>10,0,10</td>
<td>11,11,0</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

Mixed NE: (A plays 2, B plays 3 with prob $\frac{5}{11}$, C plays 5 with prob $\frac{6}{11}$)

b. Find all pure strategy and mixed strategy Nash equilibria of this game. (4 points)

No pure strategy NE.

Mixed NE? Note that for Arash, 2 strongly dominates 1.

So we have

<table>
<thead>
<tr>
<th>$p$</th>
<th>$(1-p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>10,0,10</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

$EV_{Bianca}(3) = 0 \cdot p + 11(1-p) = 11-11p$

$EV_{Bianca}(4) = 11 \cdot p + 0(1-p) = 11p$

$11-11p \geq 11p$

$p \geq \frac{11}{22} = \frac{1}{2}$

$EV_{Ciaran}(3) = 10p + 0(1-p) = 10p$

$EV_{Ciaran}(6) = 0p + 12(1-p) = 12-12p$

$(1)p = 12 - 12p$

$2p = 12 \Rightarrow p = \frac{12}{22} = \frac{6}{11}$
c. Now say that Dongsu and Estelle join the party. Dongsu can bring either 1 or 2 oranges and Estelle can bring either 1 or 2 oranges. Dongsu actually hates oranges and so Dongsu’s payoff is the negative of the total number of oranges. Estelle likes oranges but is very concerned with fairness, so Estelle’s payoff is 0 if the total number of oranges is not a multiple of 5 (because there are five people at the party and this way they can share the oranges equally). If the total number of oranges is a multiple of 5, Estelle’s payoff is the total number of oranges. Arash, Bianca, and Ciaran are the same as before.

For example, if Arash brings 1 orange, Bianca brings 4, Ciaran brings 6, Dongsu brings 1, and Estelle brings 2, then the total number of oranges is 14 and Arash’s payoff is 14, Bianca’s is 0, Ciaran’s is 14, Dongsu’s is −14, and Estelle’s is 0.

For example, if Arash brings 2 oranges, Bianca brings 4, Ciaran brings 6, Dongsu brings 1, and Estelle brings 2, then the total number of oranges is 15 and Arash’s payoff is 15, Bianca’s is 15, Ciaran’s is 0, Dongsu’s is −15, and Estelle’s is 15.

Make a prediction in this game by iteratively eliminating strongly and weakly dominated strategies, and then find all pure strategy and mixed strategy Nash equilibria of the remaining game. (Don’t write out the full five person game! Think about it first.) (4 points)

For Arash, bringing 2 oranges strictly dominates bringing 1.
For Dongsu, bringing 1 orange “ “ 2.
So we can say: A B C D E
this: 2 3 or 4 5 or 6 1 1 or 2

Estelle knows that the total number of oranges brought by everyone else is either 11, 12, or 13.
So if Estelle brings 1, the total is 12, 13, or 14. None of these numbers is a multiple of 5, so Estelle will get payoff 0 for sure.
If Estelle brings 2, the total is 13, 14, or 15, which gives Estelle at least one circumstance in which she gets payoff 1 (if the total is 15, which is a multiple of 5).
So for Estelle, 2 weakly dominates 1. So Estelle brings 2.
The only people left are Bianca and Charlie, who play this game:

\[
\begin{array}{c|c|c|c|c|c}
\text{oranges} & 0 & 1 & 2 & 3 & 4 \\
\hline
\text{Bianca} & 0 & 1 & -1 & 2 & 3 \\
\text{Charlie} & 1 & 0 & 3 & 2 & 1
\end{array}
\]
No pure strategy NE.

Mixed strategy NE?

$$EV_B(3) = 13\pi + 0(1-\pi) = 13\pi$$
$$EV_B(4) = 0\pi + 15(1-\pi) = 15 - 15\pi$$

$$13\pi = 15 - 15\pi$$
$$28\pi = 15$$
$$\pi = \frac{15}{28}$$

$$14 - 14\pi = 14\pi$$
$$28\pi = 14$$
$$\pi = \frac{14}{28} = \frac{1}{2}$$

So the mixed strategy NE is:

(A brings 3, B brings 3 with prob $\frac{1}{2}$, C brings 5 with prob $\frac{15}{28}$, D brings 4, E brings 2).