1. Say that there are three people and five candidates \( \{a, b, c, d, e\} \). Say person 1’s order of preference (from best to worst) is \( c, b, e, d, a \). Person 2’s order is \( d, c, a, b, e \). Person 3’s order is \( e, a, b, d, c \).

   a. For each candidate, find an order of voting (an “agenda”) in which that candidate wins.

   b. Say that there are three people and four candidates \( \{a, b, c, d\} \). Say person 1’s order of preference (from best to worst) is \( c, b, d, a \). Person 2’s order is \( b, a, d, c \). Person 3’s order is \( a, c, d, b \). Show that there is no order of voting in which candidate \( d \) wins. Why is this?

2. Say that eleven people vote over four candidates \( \{a, b, c, d\} \). Three people have preference order (from best to worst) \( b, a, c, d \). Three people have preference order \( b, a, d, c \). Three people have preference order \( a, c, d, b \). Two people have preference order \( a, d, c, b \).

   a. Is there a candidate which beats all others by majority rule (a Condorcet winner)?

   b. Say that instead of standard majority rule voting, that each person votes 3 points for their first choice, 2 points for their second choice, 1 point for their third choice, and no points for their last choice. This procedure is called the “Borda count.” Which candidate wins now?

   c. Which procedure (majority rule or Borda count) is more reasonable or more fair in your opinion?

3. Another system of voting is “approval voting,” in which each voter can place vote for as many candidates as she wishes: for example, one person might put one vote on candidate \( a \), and another person might put one vote on candidate \( a \) and one vote on candidate \( b \). Say we have approval voting, and each person votes for his top two candidates.

   a. Say that candidate \( a \) receives more votes by approval voting than the other two candidates. Is it possible for \( a \) to not be a Condorcet winner? If so, find an example and write down the preference orderings for each person which make this possible.

   b. Say that \( a \) is a Condorcet winner. Is it possible for \( a \) to receive fewer approval votes than some other candidate? If so, find an example and write down the preference orderings for each person which make this possible.

4. Say that there are three people who are deciding over three alternatives, \( \{a, b, c\} \). Is it possible for the Borda count winner to be different from the Condorcet winner? If so, write down the preference orderings for each person which make this possible.
5. Say that there are three groups in society. Group X’s preferences (from best to worst) are \(a, b, c\). Group Y’s preferences (from best to worst) are \(c, b, a\). Group Z’s preferences (from best to worst) are \(b, c, a\). There are 13 people in group Y and 7 people in group Z. There are \(x\) people in group X. Assume that \(x\) is an odd number (to help avoid ties).

a. Depending on the value of \(x\), what is the Condorcet winner? Is there some value of \(x\) such that there is no Condorcet winner?

b. Say society uses the Borda count system. For what values of \(x\) does the outcome of the Borda count differ from what the society would choose if they chose the Condorcet winner?

c. Say society uses approval voting in which each person votes for her top two candidates. For what values of \(x\) does the outcome of this approval voting system differ from what the society would choose if they chose the Condorcet winner?

6. Say that there are three people deciding by majority rule over four candidates \(\{a, b, c, d\}\). Person 1’s preferences (from best to worst) are \(a, b, c, d\). Person 2’s preferences (from best to worst) are \(c, d, b, a\). Person 3’s preferences (from best to worst) are \(d, a, c, b\). Consider voting agendas in which people vote on candidates sequentially.

a. Is there an agenda in which they decide on \(a\)? If there is, show it. If not, explain why.

b. Is there an agenda in which they decide on \(b\)? If there is, show it. If not, explain why.

c. Is there an agenda in which they decide on \(c\)? If there is, show it. If not, explain why.

d. Is there an agenda in which they decide on \(d\)? If there is, show it. If not, explain why.

7. Say that there are three people deciding by majority rule over eight candidates \(\{a, b, c, d, e, f, g, h\}\). Person 1’s preferences (from best to worst) are \(f, c, g, e, b, h, a, d\). Person 2’s preferences (from best to worst) are \(a, f, e, g, h, b, d, c\). Person 3’s preferences (from best to worst) are \(g, a, b, c, h, d, e, f\). Consider voting agendas in which people vote on candidates sequentially.

a. Find the top cycle.

b. For every candidate in the top cycle, find a voting agenda in which that candidate wins.
8. In 2016 the Cy Young award for best baseball pitcher in the American League was controversially awarded to Rick Porcello, even though Jason Verlander received more first-place votes. The award is given based on votes by 30 sportswriters. Each sportswriter ranks five pitchers, from first to fifth. A pitcher ranked first receives 7 points, a pitcher ranked second receives 4 points, a pitcher ranked third receives 3 points, a pitcher ranked fourth receives 2 points, and a pitcher ranked fifth receives 1 point. We don’t know each sportswriter’s preferences, but the outcome of the voting is shown in the table below.

<table>
<thead>
<tr>
<th>Pitcher</th>
<th>Team</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rick Porcello</td>
<td>Red Sox</td>
<td>8</td>
<td>18</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>137</td>
</tr>
<tr>
<td>Justin Verlander</td>
<td>Tigers</td>
<td>14</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>132</td>
</tr>
<tr>
<td>Corey Klub</td>
<td>Indians</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>1</td>
<td>98</td>
</tr>
<tr>
<td>Zach Britton</td>
<td>Orioles</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>5</td>
<td>9</td>
<td>72</td>
</tr>
<tr>
<td>Chris Sale</td>
<td>White Sox</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>6</td>
<td></td>
<td>40</td>
</tr>
<tr>
<td>J.A. Happ</td>
<td>Blue Jays</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td>14</td>
</tr>
<tr>
<td>Aaron Sanchez</td>
<td>Blue Jays</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Masahiro Tanaka</td>
<td>Yankees</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Andrew Miller</td>
<td>Yankees/Indians</td>
<td>1</td>
<td>3</td>
<td></td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Michael Fulmer</td>
<td>Tigers</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jose Quintana</td>
<td>White Sox</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that Porcello received only 8 first-place votes while Verlander received 14.

a. If instead of the 7-4-3-2-1 point system, we used the Borda count (a 4-3-2-1-0 point system), who would win the Cy Young award?

b. Under plurality voting, who would win?

c. If all you know about the sportswriters’ preferences is the table above, can you conclude that Porcello must be the Condorcet winner?

d. If all you know about the sportswriters’ preferences is the table above, is it possible for Porcello to be the Condorcet winner?

e. If all you know about the sportswriters’ preferences is the table above, can you conclude that Verlander must be the Condorcet winner?

f. If all you know about the sportswriters’ preferences is the table above, is it possible for Verlander to be the Condorcet winner?

g. If all you know about the sportswriters’ preferences is the table above, is it possible for Corey Kluber to be the Condorcet winner?

h. Can you think of a voting system which would make Corey Kluber win the Cy Young award (or at least tie for first place)?