Final exam PS 30 December 2018

This is a closed book exam. The only thing you can take into this exam is yourself and writing instruments. No calculators, computers, cell phones, headphones, audio equipment, electronics, etc. are allowed. Everything you write should be your own work. Cases of academic dishonesty will be referred to the Dean of Students office, which has the power to suspend and expel students. Partial credit will be given: math mistakes will not jeopardize your grade. Please show all steps of your work and explain what you are doing at each step. Correct answers alone are worth nothing without a clear and correct explanation of where the answers come from. Clarity and legibility are factors in the grade.

This exam has eight problems. Each question is weighted equally (12 points each). If you have a question, raise your hand and hold up the number of fingers which corresponds to the part you have questions about (if you have a question on Part 2, hold up two fingers). If you need to leave the room during the exam (to use the restroom for example), you need to sign your name on the restroom log before leaving. You can only leave the room once. When out of the room, you cannot communicate with any other person in any manner.

When the end of the exam is announced, please stop working immediately. The exams of people who continue working after the end of the exam is announced will have their scores penalized by 30 percent. Please turn in your exam to your TA. When you hand in your exam, please write your name down on the log. Please write all answers on this exam—if you write on the reverse side of pages, please indicate this clearly. Good luck!

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
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<td>1</td>
<td>5</td>
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<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
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<td>4</td>
<td>8</td>
</tr>
<tr>
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<td>total</td>
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</tbody>
</table>
Problem 1. The Beyhive (Beyoncé super fans) are determined to change the voting procedure of the Grammy awards to make Beyoncé win a Grammy. Beyoncé is competing against Cardi B, Drake, Kendrick Lamar, and Post Malone. The Grammy voters’ preferences are as follows:

<table>
<thead>
<tr>
<th></th>
<th>12 percent</th>
<th>40 percent</th>
<th>20 percent</th>
<th>28 percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Best)</td>
<td>Beyoncé</td>
<td>Post Malone</td>
<td>Drake</td>
<td>Kendrick Lamar</td>
</tr>
<tr>
<td></td>
<td>Drake</td>
<td>Kendrick Lamar</td>
<td>Drake</td>
<td>Kendrick Lamar</td>
</tr>
<tr>
<td></td>
<td>Post Malone</td>
<td>Beyoncé</td>
<td>Post Malone</td>
<td>Beyoncé</td>
</tr>
<tr>
<td>(Worst)</td>
<td>Cardi B</td>
<td>Cardi B</td>
<td>Cardi B</td>
<td>Cardi B</td>
</tr>
</tbody>
</table>

a. Is there a Condorcet winner? If so, write it down and explain why. If not, explain why not. (1 point)

There is no Condorcet winner because everyone is beaten by someone else.

b. Who is in the top cycle? (1 point)

\[ k \prec_D x \prec_D p \]

\[ \{k, D, P\} \]

c. The Beyhive somehow convinces the Grammy voters to make Jay-Z the agenda setter. Jay-Z loves Beyoncé and wants her to win. Is there a voting agenda that would make Beyoncé win? If so, write it down. If not, explain why not. (1 point)

There is no agenda which makes Beyoncé win because the only person Beyoncé beats is Cardi B, who beats no one. In other words, Beyoncé is not in the top cycle.

d. Is there a voting agenda that makes Drake win? If so, write it down. If not, explain why not. (1 point)

\[ D \prec_P k \prec_P B \prec_P C \]

This agenda makes Drake win.
The Beyhive are determined to make Beyoncé win by personally convincing $x$ number of voters from other groups to join the group that likes Beyoncé best. So voter preferences look like this:

<table>
<thead>
<tr>
<th></th>
<th>12 + 3x</th>
<th>40 - x</th>
<th>20 - x</th>
<th>28 - x</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Beyoncé</td>
<td>Post Malone</td>
<td>Drake</td>
<td>Kendrick Lamar</td>
</tr>
<tr>
<td>3</td>
<td>Drake</td>
<td>Kendrick Lamar</td>
<td>Kendrick Lamar</td>
<td>Drake</td>
</tr>
<tr>
<td>2</td>
<td>Post Malone</td>
<td>Drake</td>
<td>Post Malone</td>
<td>Beyoncé</td>
</tr>
<tr>
<td>1</td>
<td>Kendrick Lamar</td>
<td>Beyoncé</td>
<td>Beyoncé</td>
<td>Post Malone</td>
</tr>
<tr>
<td>0</td>
<td>Cardi B</td>
<td>Cardi B</td>
<td>Cardi B</td>
<td>Cardi B</td>
</tr>
</tbody>
</table>

(Best) Beyoncé

(D) Post Malone

(K) Kendrick Lamar

(P) Drake

(Video) Cardi B

Worst Cardi B

Worst Cardi B

Worst Cardi B

Worst Cardi B

By changing $x$, can Beyoncé win if the winner is decided using the Borda count? If so, what is the lowest value of $x$ that would allow her to win under the Borda count? If not, explain why not. (Assume that all ties are decided in Beyoncé’s favor. The variable $x$ can take on fractional values.) (2 points)

\[
B = 4(12+3x) + 1(40-x) + 1(20-x) + 2(28-x) = 164 + 8x
\]

\[
D = 3(12+3x) + 2(40-x) + 4(20-x) + 3(28-x) = 280
\]

\[
P = 2(12+3x) + 4(40-x) + 2(20-x) + 1(18-x) = 252 - x
\]

\[
C = 1(12+3x) + 3(40-x) + 3(20-x) + 4(28-x) = 304 - 7x
\]

\[
164 + 8x \geq 280
\]

\[8x \geq 116\]

\[x \geq \frac{116}{8} = 14.5\]

To make B win by the Borda count,

\[x \geq \frac{28}{2} = 14.5\]

f. By changing $x$, can Beyoncé win if the winner is decided using plurality voting? If so, what is the lowest value of $x$ that would allow her to win under plurality voting? If not, explain why not. (Assume that all ties are decided in Beyoncé’s favor. The variable $x$ can take on fractional values.) (2 points)

\[
B = 12+3x
\]

\[
D = 20-x
\]

\[
K = 28-x
\]

\[
P = 40-x
\]

P always leads D and K in plurality voting.

So we check

\[
12+3x \geq 40-x
\]

\[4x \geq 28\]

\[x \geq 7\]

So to make B win by plurality, $x \geq 7$. 
Here are the preferences again.

<table>
<thead>
<tr>
<th>Preferences</th>
<th>12 + 3x</th>
<th>40 − x</th>
<th>20 − x</th>
<th>28 − x</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(Best)</strong></td>
<td>Beyoncé</td>
<td>Post Malone</td>
<td>Drake</td>
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<td>Kendrick Lamar</td>
<td>Drake</td>
<td></td>
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<tr>
<td>Post Malone</td>
<td>Kendrick Lamar</td>
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<td>Post Malone</td>
<td></td>
</tr>
<tr>
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<td>Beyoncé</td>
<td>Cardi B</td>
<td>Cardi B</td>
<td></td>
</tr>
<tr>
<td><strong>(Worst)</strong></td>
<td>Cardi B</td>
<td>Cardi B</td>
<td>Cardi B</td>
<td>Cardi B</td>
</tr>
</tbody>
</table>

g. By changing $x$, can Beyoncé win if the winner is decided using approval voting, where each voter votes for their top two choices? If so, what is the lowest value of $x$ that would allow her to win under approval voting? If not, explain why not. (Assume that all ties are decided in Beyoncé’s favor. The variable $x$ can take on fractional values.) (2 points)

Note that Drake will always have more approval vote points than Beyoncé because anytime Beyoncé gets one from this group, Drake gets one also, and Drake gets additional points from the other groups. So it is not possible for Beyoncé to win under approval voting, no matter how big $x$ is.

h. By changing $x$, can Beyoncé win if the winner is decided using a runoff system? If so, what is the lowest value of $x$ that would allow her to win under a runoff system? If not, explain why not. (Assume that all ties are decided in Beyoncé’s favor. The variable $x$ can take on fractional values.) (2 points)

To get into the runoff election, B must be at least second place among first-choice votes. The second-highest vote getter among B’s opponents is $k$. So we must have

$$12 + 3x + 28 - x \geq 40 - x + 20 - x$$

B will go up against P in the runoff election. So we must have

$$12 + 3x + 28 - x \geq 40 - x + 20 - x$$

$$40 + 2x \geq 60 - 2x \Rightarrow 4x \geq 20 \Rightarrow x \geq 5$$

Since we need to have both $x \geq 4$ and $x \geq 5$, we must have $x \geq 5$ to make B win.
Problem 2. Say that Facebook, Google, and Huawei are each looking to hire a new software engineer. The firms have preferences over the three engineers on the market, persons X, Y, and Z. The engineers also have preferences about where they want to work, as shown below.

<table>
<thead>
<tr>
<th>F</th>
<th>G</th>
<th>H</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
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</tr>
</tbody>
</table>

(Best) Z Z X (Best) G G F
X Y Y H H H
(Worst) Y X Z (Worst) F F F

For example, Facebook likes engineer Z best, X second-best, and Y least. Engineer Y likes Google best, Huawei second-best, and Facebook least.

a. Find all stable matchings. (2 points)

\[
\begin{align*}
\text{Firms ask:} & \\
(FZ, GZ, HX) & \quad \text{(FZ, GZ, HX) stable} \\
(FX, HY, HX) & \\
(FY, C2, HX) & \quad \text{(FY, C2, HX) stable} \\
\end{align*}
\]

\[
\text{Engineers ask:} \quad (X6, Y6, ZF) \quad (X6, Y6, ZF) \quad \text{stable}
\]

So the only stable match is \((HX, Y6, ZF)\).

b. Now say that there is a scandal at Facebook (it has been compromised by the Russian government) and thus all engineers don’t want to work at Facebook and it falls to last place in all the engineers’ rankings. The engineers’ preferences about the other companies are the same as in the original situation in part a. The companies’ preferences are the same as in the original situation in part a. Hence preferences look like this. Find all stable matchings. (2 points)

<table>
<thead>
<tr>
<th>F</th>
<th>G</th>
<th>H</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>-</td>
</tr>
</tbody>
</table>

(Best) Z Z X (Best) G G G
X Y Y H H H
(Worst) Y X Z (Worst) F F F

\[
\begin{align*}
\text{Firms ask:} & \\
(FZ, GZ, HX) & \quad \text{(FZ, GZ, HX) stable} \\
(FX, HY, HX) & \\
(FY, C2, HX) & \quad \text{(FY, C2, HX) stable} \\
\end{align*}
\]

\[
\text{Engineers ask:} \quad (X6, Y6, ZC) \quad (X6, Y6, ZC) \quad (XH, YH, ZH) \quad (XH, YF, ZC) \quad \text{stable}
\]

So the only stable match is \((XH, YF, ZC)\).
c. Now say that instead there is a scandal at Google (it has been compromised by the Russian government) and thus all engineers don’t want to work at Google and it falls to last place in all the engineers’ rankings. The engineers’ preferences about the other companies are the same as in the original situation in part a. The companies’ preferences are the same as in the original situation in part a. Find all stable matchings. (2 points)

\[
\begin{array}{ccc}
F & Y & H \\
C & F & X \\
X & Y & F \\
Y & X & H \\
\end{array}
\]

\[
\begin{array}{ccc}
F & Y & Z \\
Z & Z & X \\
X & Y & Y \\
Y & X & Z \\
\end{array}
\]

Final asks:  
\[(FZ, GZ, HX)\]  
\[(FZ, GY, HX)\] stable

Engineer asks:  
\[(XH, YH, ZF)\]  
\[(XH, YF, ZF)\]  
\[(XH, Y6, ZF)\] stable

Same

So only stable match is \((XH, Y6, ZF)\).

d. Now say that instead there is a scandal at Huawei (it has been compromised by the Russian government) and thus all engineers don’t want to work at Huawei and it falls to last place in all the engineers’ rankings. The engineers’ preferences about the other companies are the same as in the original situation in part a. The companies’ preferences are the same as in the original situation in part a. Find all stable matchings. (2 points)

\[
\begin{array}{ccc}
F & C & H \\
C & X & Y \\
X & Y & F \\
Y & X & Z \\
\end{array}
\]

\[
\begin{array}{ccc}
F & Y & Z \\
Z & Z & X \\
X & Y & Y \\
Y & X & Z \\
\end{array}
\]

Final asks:  
\[(FZ, GZ, HX)\]  
\[(FZ, GY, HX)\] stable

Engineer asks:  
\[(X6, Y6, ZF)\]  
\[(XF, Y6, ZF)\]  
\[(XH, Y6, ZF)\] stable

Same

So the only stable match is \((XH, Y6, ZF)\).
e. Is it possible for one company to gain when one of its competitors has a scandal? In other words, is it possible for one company to prefer situations b., c., or d. over the original situation a.? If so, write down which company gains from which scandal. If not, explain why not. (1 point)

When there is a scandal at Facebook, the match goes from \((Y, X, Y, G, F)\) to \((X, Y, G, F)\).

Google gains from this because it gets its favorite engineer, \(Z\), instead of \(Y\).

f. Now say that engineer Z is friends with Perez Hilton and can create a scandal at any company: Facebook, Google, or Huawei. As before, by creating a scandal at a company, engineer Z can make all engineers (including engineer Z themself) like that company least. The companies’ preferences are shown below. Note that the preferences of X and Y are not known. By revealing the information and causing a scandal, engineer Z hopes to get a better job. Can Z get a better job by creating a scandal? If so, write down a specific example which shows how Z can get a better job by creating a scandal. If not, explain why not. (3 points)

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<tr>
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<th>F</th>
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<th>H</th>
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</thead>
<tbody>
<tr>
<td>(Best)</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>(Worst)</td>
<td>Z</td>
<td>Z</td>
<td>Z</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Best)</td>
<td>H</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Worst)</td>
<td>F</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

All firms like \(Y\) best. So \(Y\) will be matched with its first choice.

The remaining two firms both like \(X\) best (because \(Y\) is taken).

Hence \(X\) will be matched with whichever of the two firms \(X\) likes best.

If there is a scandal, the firm which has the scandal will be rejected by both \(X\) and \(Y\). The firm with the scandal will then be matched with \(Z\), the only engineer left.

But \(Z\) likes the firm with the scandal least also.

So by causing the scandal, \(Z\) will get \(Z\)'s worst choice.

So \(Z\) cannot gain by creating a scandal.
Problem 3. Consider the following game.

a. Write down a subgame perfect Nash equilibrium of this game (if there is more than one, you do not have to write down all of them). (2 points)

b. Represent this as a strategic form game. (2 points)

\[
\begin{array}{cc|cc|cc}
& d & f & g & e & s \\
\hline
a & 2, 1 & 2, 1 & 4, 4 & 4, 4 & \\
b & 2, 1 & 2, 1 & 4, 4 & 4, 4 & \\
c & 8, 6 & 0, 7 & 8, 6 & 0, 7 & \\
d & 8, 6 & 4, 3 & 8, 6 & 4, 3 & \\
e & 5, 0 & 5, 0 & 5, 0 & 5, 0 & \\
f & 5, 0 & 5, 0 & 5, 0 & 5, 0 & \\
g & 5, 0 & 5, 0 & 5, 0 & 5, 0 & \\
r & 5, 0 & 5, 0 & 5, 0 & 5, 0 & \\
\end{array}
\]

b. Use the method of iterative elimination of (strongly and weakly) dominated strategies to make a prediction in the strategic form game. Try to eliminate as much as possible. Please show the order of elimination. (2 points)

1. $ef$ w. dom $df$
2. $eg$ w. dom $dg$
3. $bs$ w. dom $br$
4. $et$ w. dom $eg$
5. $bs$ s. dom $ar, as, cr$ and $cs$. (other solutions are possible)
Here is the game again with a small change. Note that the payoff at the bottom is now $k, 0$.

d. The subgame perfect Nash equilibrium of this game, and the set of all pure strategy Nash equilibria, now depends on $k$. Please fill out the table below. For each value of $k$, write down the subgame perfect Nash equilibrium, and write down all pure strategy Nash equilibria. Note that $k = 5$ corresponds to the game in part a. (6 points)

<table>
<thead>
<tr>
<th>$k$</th>
<th>SPNE</th>
<th>Pure Strategy NE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(bs, df)$</td>
<td>$(bs, df)$, $(bs, dg)$, $(bs, ef)$, $(cr, eg)$</td>
</tr>
<tr>
<td>3</td>
<td>/ /</td>
<td>/ /</td>
</tr>
<tr>
<td>5</td>
<td>/ /</td>
<td>$(bs, df)$, $(cr, dg)$, $(bs, ef)$, $(cs, eg)$</td>
</tr>
<tr>
<td>7</td>
<td>/ /</td>
<td>/ /</td>
</tr>
<tr>
<td>9</td>
<td>$(cs, ef)$</td>
<td>$(cr, df)$, $(cr, dg)$, $(cr, ef)$, $(cr, eg)$</td>
</tr>
</tbody>
</table>
Problem 4. Say that there are three voters deciding over three candidates, A, B, and C.

a. Is it possible to fill in the blanks below so that A is the Condorcet winner? If so, fill in the blanks below and explain why. If not, explain why not. (2 points)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
(Best) & B & A & [A] \\
& A & C & B \\
(Worst) & C & B & [C] \\
\end{array}
\]

Here \( A > B, A > C \) since a majority likes A the best. So A is the Condorcet winner.

b. Is it possible to fill in the blanks below so that there is no Condorcet winner? If so, fill in the blanks below and explain why. If not, explain why not. (2 points)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
(Best) & B & A & [C] \\
& A & C & B \\
(Worst) & C & B & [A] \\
\end{array}
\]

Here \( A > C, C > B, B > A \). So there is no Condorcet winner.
Again, there are three people deciding over three candidates, A, B, and C.

c. Is it possible to fill in the blanks below so that A is the Condorcet winner? If so, fill in the blanks below and explain why. If not, explain why not. (2 points)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Best)</td>
<td>[ ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>(Worst)</td>
<td>[ ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
</tbody>
</table>

In this row, there must be at least two B or at least two C.

If there are at least two B, then B is the Condorcet winner.

If there are at least two C, then C is the Condorcet winner.

So it is not possible for A to be the Condorcet winner.

d. Is it possible to fill in the blanks below so that there is no Condorcet winner? If so, fill in the blanks below and explain why. If not, explain why not. (2 points)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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</tr>
</thead>
<tbody>
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<td>[ ]</td>
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</tr>
<tr>
<td></td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>(Worst)</td>
<td>[ ]</td>
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</tr>
</tbody>
</table>

By the reasoning above, either B or C must be the Condorcet winner.

It is not possible for there to be no Condorcet winner.
Now say that there are three people deciding over four candidates, A, B, C, and D.

e. Is it possible to fill in the blanks below so that A is the Condorcet winner? If so, fill in the blanks below and explain why. If not, explain why not. (2 points)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Best)</td>
<td>[D]</td>
<td>[C]</td>
<td>[B]</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>(Worst)</td>
<td>[C]</td>
<td>[D]</td>
<td>[D]</td>
</tr>
</tbody>
</table>

Yes, it is possible.

f. Is it possible to fill in the blanks below so that there is no Condorcet winner? If so, fill in the blanks below and explain why. If not, explain why not. (2 points)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
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<td>[ ]</td>
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</tr>
<tr>
<td></td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>(Worst)</td>
<td>[ ]</td>
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<td>[ ]</td>
</tr>
</tbody>
</table>

So, the only way that B, C, or D is not the CW is if the top row contains exactly one B, one C, and one D. But then the bottom two rows contain 2 Bs, 2 Cs, and 2 Ds, which makes A the CW, as in part e., above. So it is not possible for there to be no CW.
Problem 5. A climate agreement needs the support of 40 countries in order to pass. These 40 countries are in three groups. The first group is the Island Countries, which is composed of 15 countries. Each member of this group has a threshold of 10; in other words, a country in this group will support the agreement if at least 10 other countries in the world support the agreement. The second group is the Coastal Countries, which is composed of 20 countries. Each member of this group has a threshold of 20. Finally, the third group is the Mountain Countries, which is composed of 5 countries. Each member of this group has a threshold of 30.

a. What are the pure strategy Nash equilibria of this game? (3 points)

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>10</th>
<th>20</th>
<th>20</th>
<th>30</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Y</td>
<td>N</td>
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</table>

b. Say that the US can bribe countries. By giving $1 billion to a country, the US can make a country’s threshold increase by 10. For example, if the US gives $1 billion to an Island country, its threshold will be 20. If the US gives $2 billion to an Island country, its threshold will be 30. The US can only spend multiples of $1 billion and cannot spend fractions of billions (for example, the US cannot spend a half billion dollars to make a country’s threshold increase by 5). The US wants to guarantee that the agreement will fail. In other words, the US wants to change thresholds so that there is no Nash equilibrium in which the agreement passes. How does the US achieve its goal by spending the least amount of money? Please explain your work. (3 points)

The US can bribe a single M country, thereby making its threshold 40 and hence ensuring that it will never support the agreement.

Since the agreement passes only if it gets the support of all 40, the agreement will not pass.
c. Now say that the US wants to ensure that the agreement succeeds. By giving $1 billion of aid to a country, the US can make a country’s threshold decrease by 10. The US wants to change thresholds so that in all Nash equilibria, the agreement passes. How does the US achieve its goal by spending the least amount of money? Please explain your work. (3 points)

<table>
<thead>
<tr>
<th>T</th>
<th>10</th>
<th>→ 0</th>
<th>ten people</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>20</td>
<td>→ 10</td>
<td>five people</td>
</tr>
<tr>
<td>M</td>
<td>30</td>
<td>→ 10</td>
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To make sure that we don’t have a NE in which no one participated, we need to make some people have threshold 0. Once ten people have threshold 0, the other 10 countries join in. Then we need to make five C countries have threshold 10 to make sure they join in. Once 20 countries participate, the remaining C countries join in and then the M countries join in.

d. Now say that the US wants to create maximum deadlock and wants exactly 20 countries to support the agreement and 20 countries to not support it. By giving $1 billion of aid to a country, the US can make a country’s threshold decrease by 10 or increase by 10. The US wants to change thresholds so that in all Nash equilibria, exactly 20 countries support the agreement and 20 do not support it. How does the US achieve its goal by spending the least amount of money? Please explain your work. (3 points)

<table>
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To guarantee that at least 20 participate, you need to make these people have threshold 0. Then these people will surely participate. You need to change five of the C countries to threshold 10 to make sure they participate. You need to raise these thresholds to keep them from jumping in when 20 people participate.

You need to make 10 people have threshold 40 because otherwise there would be a NE in which 30 people participate.
Problem 6. Say that there are 30 voters. Six are at HL (Hard Left), six are at ML (Moderate Left), six are at C (Center), six are at MR (Moderate Right), and six are at HR (Hard Right), as shown below.

<table>
<thead>
<tr>
<th></th>
<th>HL</th>
<th>ML</th>
<th>C</th>
<th>MR</th>
<th>HR</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6</td>
<td>6</td>
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</tbody>
</table>

a. There are two candidates who are looking for votes. Each candidate takes a position, either HL, ML, C, MR, or HR. A voter votes for the candidate whose position is closest to the voter’s own position. If two candidates are equally far away from voters at a given position, then the two candidates split the votes at that position equally. For example, if candidate 1 takes position HL and candidate 2 takes position C, then half of the voters at ML vote for candidate 1 and half of the voters at ML vote for candidate 2. So if candidate 1 takes position HL and candidate 2 takes position C, then candidate 1 gets a total of 9 votes and candidate 2 gets a total of 21 votes.

However, if the two candidates take exactly the same position, then voters do not get excited about either candidate (because there is no meaningful difference between them) and no one votes at all.

b. Model this as a strategic form game in which each candidate’s payoff is the number of votes the candidate gets. (3 points)

c. Find all pure strategy Nash equilibria of this game. (3 points)

\[ \text{NE: (C, ML), (C, MR), (ML, C), (MR, C)} \]
Here are the voters again.

<table>
<thead>
<tr>
<th></th>
<th>HL</th>
<th>ML</th>
<th>C</th>
<th>MR</th>
<th>HR</th>
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<td></td>
<td>6</td>
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</tbody>
</table>

Now say that there are three candidates. Again, each candidate takes a position, either HL, ML, C, MR, or HR. A voter votes for the candidate whose position is closest to the voter’s own position. If two or more candidates are equally far away from voters at a given position, then the two or more candidates split the votes at that position equally.

If a candidate takes exactly the same position as another candidate, then the candidate gets zero votes. In other words, the only way a candidate can get nonzero votes is if the candidate takes a position which no other candidate takes. If two candidates take the same position and the third takes a different position, then the candidate taking the unique position gets all the votes. If all three candidates take the same position, then all candidates get zero votes.

d. Find the pure strategy Nash equilibria of this game. If you find a lot of them, it is not necessary to tediously write down all of them—just write down some of them and explain what the set of Nash equilibria consists of. Also, don’t write out the entire strategic form game! Just think about it. (6 points)
Problem 7. Say that Ary, Ben, and Celia are raising money for a scholarship fund. Each person makes a pledge of how much they will donate. Ary can either pledge 1 or 0. Ben can either pledge 1 or 0. Celia is a little richer and can either pledge 2 or 0. The scholarship fund succeeds if the total amount of pledges is 3 or more. If the scholarship fund succeeds, then it is worth 5 to everyone, and also everyone has to contribute the amount they pledged. If the scholarship fund does not succeed, then no one has to pay anything and everyone gets 0. For example, if everyone pledges, then Ary and Ben get payoff 4 and Celia gets payoff 3.

a. Say that everyone pledges simultaneously. Model this as a strategic form game. (2 points)

\[
\begin{array}{ccc}
A & B & C \\
\hline
1 & 4; 4, 3 & 4; 5, 3 \\
0 & 5; 4, 3 & 0; 0, 0 \\
2 & 0 & 0; 0, 0
\end{array}
\]

b. Simplify the game by iteratively eliminating strongly or weakly dominated strategies. Find all pure strategy and mixed strategy Nash equilibria of the remaining game. (3 points)

For C, 2 w. dominates 0, so we have

\[
\begin{array}{ccc}
A & B & C \\
\hline
1 & 4; 4, 3 & 4; 5, 3 \\
0 & 5; 4, 3 & 0; 0, 0
\end{array}
\]

Pure strategy NE: (0, 1, 2) (1, 0, 2)

Mixed strategy NE?

For Ary:

\[
E_{UA}(0) = 42 + 4(-1) = 4 \\
\]

For Ary:

\[
E_{UA}(1) = 52 + 0(-1) = 5 \\
\]

For Celia:

\[
E_{UC}(0) = 4(1 \times p) + 4(1 \times (1-p)) = 4 \\
\]

For Ben:

\[
E_{UB}(0) = 5(1 \times p) + 0(1 \times (1-p)) = 5 \\
\]

Mixed NE:

\[
(A : p_{B} = \frac{5}{3} \text{, } p_{A} = \frac{1}{3} \text{, } p_{C} = 1) \quad (B : p_{A} = \frac{5}{3} \text{, } p_{B} = \frac{1}{3} \text{, } p_{C} = 2) \quad (C : p_{A} = \frac{1}{3} \text{, } p_{B} = \frac{5}{3} \text{, } p_{C} = 0)
\]
c. Now say that they pledge in sequence. First Ary pledges 1 or 0, then Ben pledges 1 or 0, and finally Celia pledges 2 or 0. Model this as an extensive form game. (2 points)

d. Write down a subgame perfect Nash equilibrium of this game (it is sufficient to do this by writing arrows in the tree). If there is more than one, you don’t have to write down all of them. (2 points)

One of the SPNE is shown in the arrows above. There are other SPNE because of ties.
e. Now say that there are seven donors: Ary, Ben, Celia, Deepa, Emir, Faruk, and Golda. First Ary pledges 1 or 0, then Ben pledges 1 or 0, then Celia, and so forth. Everyone except Celia pledges either 1 or 0. As before, Celia pledges either 2 or 0. As before, the scholarship fund succeeds if the total amount of pledges is 3 or more. As before, if the scholarship fund succeeds, it is worth 5 to everyone, and also everyone has to contribute the amount they pledged. If the scholarship fund does not succeed, then no one has to pay anything and everyone gets 0.

In the subgame perfect Nash equilibrium of this game, who ends up pledging a positive amount? Please explain why. You can solve this problem without writing out the entire extensive form game, which would be huge. (3 points)

If the total amount I pledge is 2 when it is Golda’s turn to decide, then Golda will pledge 1 and get a payoff of 4 instead of 0.

If the total amount I pledge is 1 when it is Faruk’s turn to decide, Faruk will pledge 1 knowing that Golda will therefore pledge 1 (by pledging 1, Faruk gets a payoff of 4 and by pledging 0, Faruk gets a payoff of 0).

If the total amount I pledge is 0 when it is Emir’s turn to decide, then Emir will pledge 1 knowing that Faruk will therefore pledge 1 and Golda will pledge 1.

Ary, Ben, Celia, Deepa all know this and hence pledge 0 knowing that Emir, Faruk, and Golda will pick up the slack and fund the scholarship.

So in a SPNE, Emir, Faruk, and Golda end up pledging a positive amount.
Problem 8. Say that three people are deciding among seven candidates, A, B, C, D, E, F, and G. Their preferences are shown below.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>G</td>
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<tr>
<td>D</td>
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<td>C</td>
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<td>B</td>
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<tr>
<td>E</td>
<td>G</td>
<td>A</td>
</tr>
</tbody>
</table>

(Worst) F A F

a. Is there a voting agenda in which the group chooses G? If so, write it down as a tree. If not, explain why not. (2 points)

b. Is there a voting agenda in which the group chooses A? If so, write it down as a tree. If not, explain why not. (2 points)

No, because the only candidate A beats is F, and F does not beat anyone

c. What is the top cycle? (4 points)

So the top cycle is {B, C, D, E, G}
d. Say that a political consultant goes around to the candidates and offers a service. This consultant is able to persuade any candidate, but only one candidate, to drop out. In other words, a candidate who hires this consultant can make one opponent drop out. Which candidate can benefit the most by hiring the consultant? Which candidate will the consultant make drop out? Please explain your work. (4 points)

**Note that C has only one opponent who can beat C. All other candidates are beaten by two or more others.**

So if C can make this one opponent (D) drop out, C will be the Condorcet winner.

So C has the most to gain from hiring the consultant, and will ask the consultant to persuade D to drop out.